Pre-Algebra Notes

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Answer Key

This resource is pre-release and is still a work in progress. All contents are subject to change.

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1.1 Integers and Absolute Value

The natural numbers are the numbers you can count to, starting from one.

$$1, 2, 3, 4, 5, \ldots$$

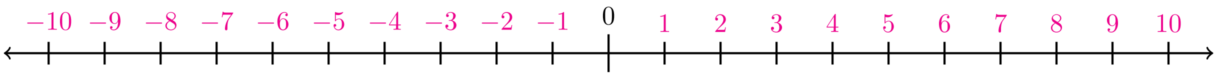
The whole numbers are the numbers you can count to, starting from zero.

$$0, 1, 2, 3, 4, 5, \ldots$$

The integers are the numbers you can count to, but you're also allowed to count backwards. This means the integers include the natural numbers and their negatives, as well as zero.

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

A positive number is any number greater than zero. A negative number is any number less than zero. A negative sign in front of a number means that it has the opposite direction on a number line.



The absolute value of a number is the distance of a number from zero on a number line. The symbol for absolute value is vertical lines either side of a number.

Example

Evaluate each of the absolute value expressions.

$|7| = 7$ $|-7| = 7$ $|-4| = 4$ $|9| = 9$

We can use the symbols < (less than), > (greater than), and = (equals) to show the order of numbers. On a number line, lesser numbers are to the left, and greater numbers are to the right.

Example

Write =, < or > to correctly indicate the order of each pair of integers.

$9 > 2$ $-4 < 1$ $3 > -8$

$5 = |-5|$ $-7 < -2$ $|8| = 8$

1.2 Integer Operations

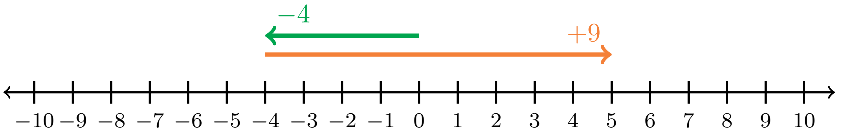
The sum of a set of numbers is the result of their addition.

The additive identity is zero, because its sum with any other number is the other number. A positive number and its negative are each the additive inverse (or opposite) of the other because they sum to zero.

Example

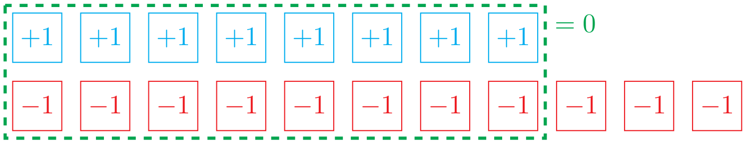
Use the number line to evaluate the sum.

$-4 + 9 = 5$



Use tiles to evaluate the sum.

$8 + (-11) = -3$

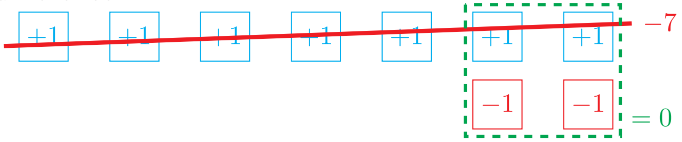


The difference of two numbers is the result of their subtraction, which is the inverse of addition. This means we can subtract a number by adding its opposite.

Example

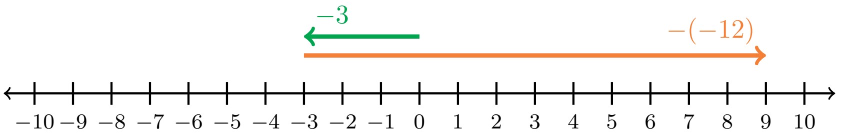
Use tiles to evaluate the difference.

$5 - 7 = -2$



Use the number line to evaluate the difference.

$-3 - (-12) = 9$



Write each difference as a sum. Then evaluate them.

$6 - (-9) = 6 + 9$
$= 15$

$-8 - (-4) = -8 + 4$
$= -4$

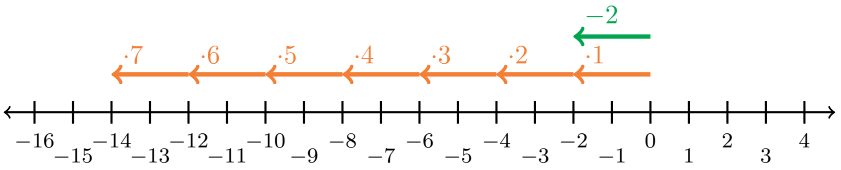
$-5 - (-11) = -5 + 11$
$= 6$

The product of a set of numbers is the result of their multiplication, which represents repeated addition. For two factors, one factor counts how many times the other factor is added.

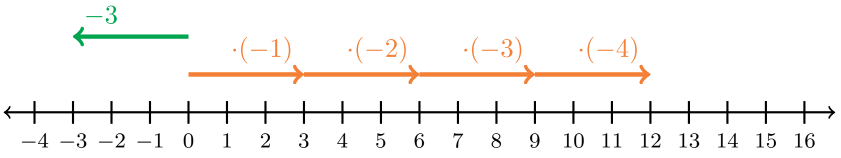
Example

Use the number line to evaluate each product.

$(-2) \cdot 7 = -14$



$(-3) \cdot (-4) = 12$

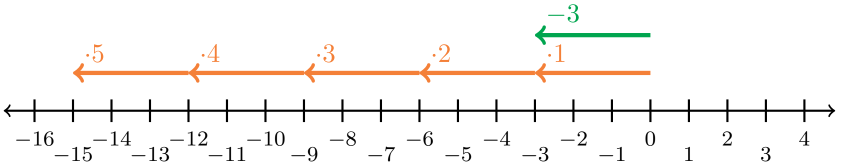


The quotient of two numbers is the result of their division, which is the inverse of multiplying. It asks what to multiply the divisor (second number) by to get the dividend (first number).

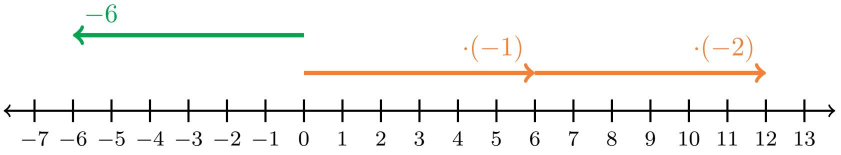
Example

Use the number line to evaluate each quotient.

$$\frac{-15}{-3} = 5$$



$$\frac{12}{-6} = -2$$



Notice that multiplying or dividing by a negative reverses the sign (or direction) of the result. Therefore, the product or quotient of two negative numbers is positive.

Example

Evaluate each product and quotient.

$5 \cdot 7 = 35$

$(-6) \cdot 9 = -54$

$8 \cdot (-4) = -32$

$(-11) \cdot (-12) = 132$

$\frac{56}{8} = 7$

$\frac{-91}{7} = -13$

$\frac{64}{-4} = -16$

$\frac{-42}{-14} = 3$

1.3 Rational Numbers

A fraction is a number written as the ratio (quotient, division) of two numbers. It contains a numerator on the top and a denominator on the bottom.

A rational number is a number which can be written as a fraction using integers.

Example

Write each as a fraction to show that it is a rational number.

$-19 = \frac{-19}{1}$ $2.8 = \frac{14}{5}$ $0.\overline{3} = \frac{1}{3}$

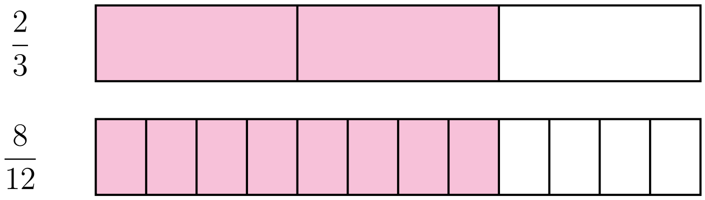
In general:

- All integers are rational.
- All terminating decimals are rational.
- All repeating decimals are rational.

Fractions are equivalent if they represent the same number.

Example

Use the fraction bars to show that $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent.



$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}$$

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

A fraction can be simplified by dividing both the numerator and denominator by their greatest common factor.

Example

Simplify each of the following fractions.

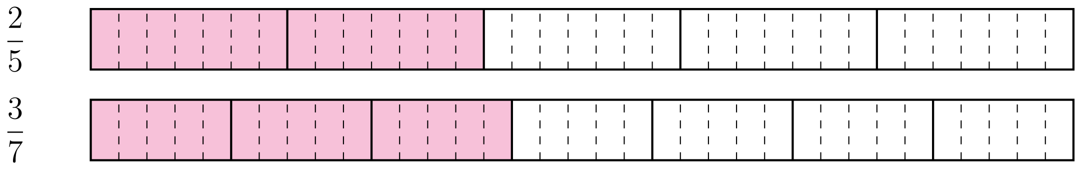
$$\frac{10}{35} = \frac{10 \div 5}{35 \div 5} = \frac{2}{7}$$

$$\frac{20}{32} = \frac{20 \div 4}{32 \div 4} = \frac{5}{8}$$

Fractions with different denominators are difficult to order and compare, so its useful to write them with a common denominator. The least common denominator, which is the least common multiple of the denominators, is preferred.

Example

Which is greater of $\frac{2}{5}$ and $\frac{3}{7}$?



$\frac{2}{5} = \frac{14}{35}$ and $\frac{3}{7} = \frac{15}{35}$, so $\frac{2}{5} < \frac{3}{7}$.

Write $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{2}{3}$ in ascending (least to greatest) order.

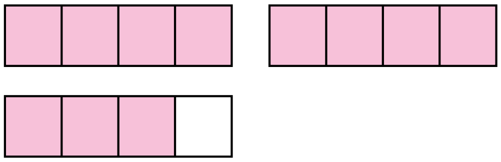
The LCM of 4, 6, and 3 is 12.

$\frac{3}{4} = \frac{9}{12}$ $\frac{5}{6} = \frac{10}{12}$ $\frac{2}{3} = \frac{8}{12}$ The order is $\frac{2}{3} < \frac{3}{4} < \frac{5}{6}$.

A proper fraction has a numerator less than the denominator, and is valued between zero and one. A fraction greater than one can be written as a mixed number, as the sum of an integer and a proper fraction; or as an improper fraction, with a numerator greater than the denominator.

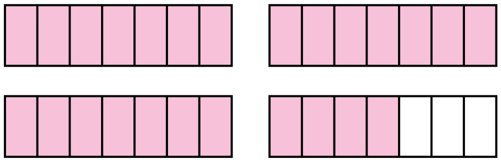
Example

Write the mixed number $2\frac{3}{4}$ as an improper fraction.



$$2\frac{3}{4} = \frac{8+3}{4}$$
$$= \frac{11}{8}$$

Write the improper fraction $\frac{25}{7}$ as an mixed number.



$$\frac{25}{7} = \frac{21+4}{7}$$
$$= 3\frac{4}{7}$$

1.4 Adding and Subtracting Fractions

Fractions can be added or subtracted as long as they have a common denominator, by adding or subtracting the numerators and keeping the same denominator.

Example

Evaluate each of the following.

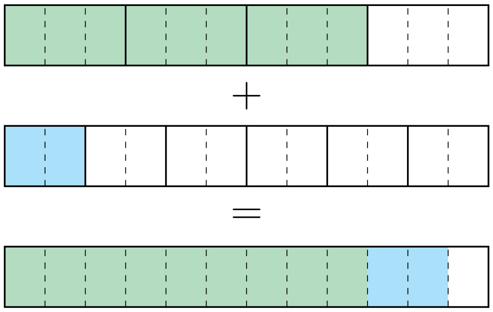
$$\frac{5}{7}+\frac{4}{7}=\frac{9}{7} = 1\frac{2}{7}$$

$$\frac{1}{4}-\frac{3}{4}=-\frac{2}{4} = -\frac{1}{2}$$

$$\frac{1}{10}-\frac{7}{10}+\frac{9}{10}=-\frac{6}{10}+\frac{9}{10} = \frac{3}{10}$$

Example

Use the fraction bars to represent $\frac{3}{4}$ and $\frac{1}{6}$. Then find the sum of the fractions.



The LCM of 4 and 6 is 12.

$$\frac{3}{4}+\frac{1}{6}=\frac{9}{12}+\frac{2}{12} = \frac{11}{12}$$

Example

Evaluate each of the following.

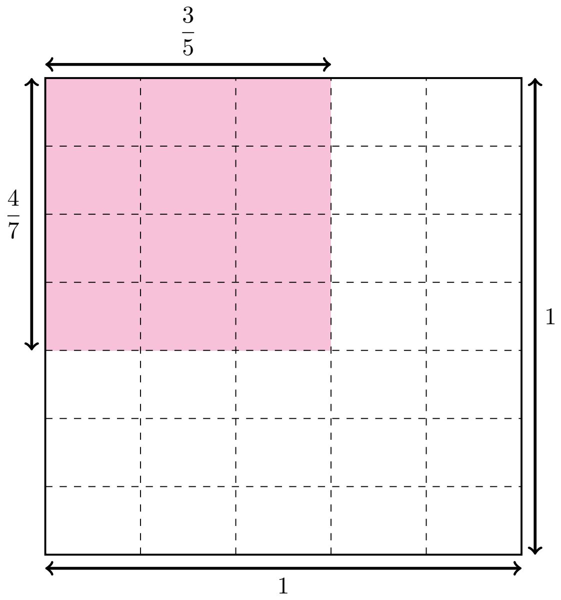
$$\frac{9}{10}-\frac{18}{25}=\frac{45}{40}-\frac{36}{50} = \frac{9}{50}$$

$$\frac{2}{3}+\frac{4}{5}=\frac{10}{15}+\frac{12}{15} = \frac{22}{15} = 1\frac{7}{15}$$

$$2\frac{5}{8}-4\frac{1}{4}=\frac{21}{8}-\frac{17}{4} = \frac{21}{8}-\frac{34}{8} = -\frac{13}{8} = -1\frac{5}{8}$$

1.5 Multiplying and Dividing Fractions

Example



Shade the region with dimensions $\frac{3}{5} \times \frac{4}{7}$.

How many equally sized sections make the 1 unit square?

$$5 \times 7 = 35$$

How many equally sized sections are in the shaded region?

$$3 \times 4 = 12$$

What fraction of the 1 unit square is shaded?

$$\frac{3}{5} \times \frac{4}{7} = \frac{12}{35}$$

To multiply fractions, multiply the numerators to get the resulting numerator, and multiply the denominators to get the resulting denominator.

If multiplying an integer by a fraction, write it as a fraction with one for the denominator.

If multiplying a mixed number, write it as an improper fraction first.

Example

Evaluate each product.

$$\frac{4}{9} \cdot \frac{3}{8} = \frac{12}{72} = \frac{1}{6}$$

$$2\frac{4}{5} \times \frac{1}{7} = \frac{14}{5} \times \frac{1}{7} = \frac{14}{35} = \frac{2}{5}$$

$$\left(-\frac{3}{10}\right)\left(\frac{20}{9}\right) = -\frac{60}{90} = -\frac{2}{3}$$

The multiplicative identity is one because its product with any other number is the other number. The reciprocal (or multiplicative inverse) of a number is another number which multiplies it to result in one.

Example

Show that these numbers are reciprocals.

$\frac{5}{6}$ and $\frac{6}{5}$

$$\frac{5}{6}\cdot\frac{6}{5}=\frac{30}{30}$$
$$=1$$

$\frac{1}{7}$ and 7

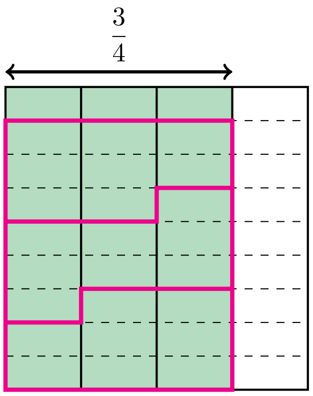
$$\frac{1}{7}\cdot\frac{7}{1}=\frac{7}{7}$$
$$=1$$

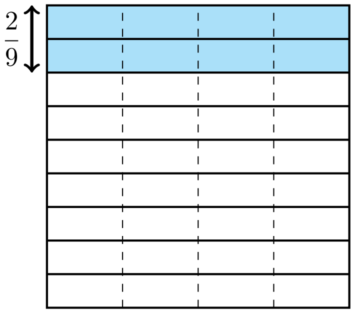
$1\frac{1}{2}$ and $\frac{2}{3}$

$$\frac{3}{2}\cdot\frac{2}{3}=\frac{6}{6}$$
$$=1$$

The reciprocal of a proper or improper fraction can be found by switching the numerator and denominator.

Example





Shade the regions showing $\frac{3}{4}$ and $\frac{2}{9}$.

How many small sections make $\frac{3}{4}$?

$$3\times 9=27$$

How many small sections make $\frac{2}{9}$?

$$4\times 2=8$$

How many times does $\frac{2}{9}$ fit into $\frac{3}{4}$?

$$\frac{3}{4}\div\frac{2}{9}=\frac{3}{4}\cdot\frac{9}{2}=\frac{27}{8}=3\frac{3}{8}$$

Dividing by a number is equivalent to multiplying by its reciprocal.

Example

Evaluate each quotient.

$$\frac{5}{4}\div\frac{7}{8}=\frac{5}{4}\cdot\frac{8}{7}$$
$$=\frac{40}{28}$$
$$=\frac{10}{7}$$

$$\frac{3}{4}\div 6=\frac{3}{4}\cdot\frac{1}{6}$$
$$=\frac{3}{24}$$
$$=\frac{1}{8}$$

$$\frac{2}{3}\div\left(-\frac{6}{11}\right)=\frac{2}{3}\cdot\left(-\frac{11}{6}\right)$$
$$=-\frac{22}{18}$$
$$=-\frac{11}{9}$$

$$2\frac{1}{3}\div 3\frac{2}{5}=\frac{7}{3}\div\frac{17}{5}$$
$$=\frac{7}{3}\cdot\frac{5}{17}$$
$$=\frac{35}{51}$$

$$9\div\frac{3}{4}=\frac{9}{1}\cdot\frac{4}{3}$$
$$=\frac{36}{3}$$
$$=12$$

$$-2\frac{1}{5}\div(-3)=-\frac{11}{5}\cdot\left(-\frac{1}{3}\right)$$
$$=\frac{11}{15}$$

1.6 Rational Number Equivalents

Decimals and Percents

"Percent" literally means to divide by 100, so 100% is equal to $\frac{100}{100} = 1$.

- Convert percent to decimal: divide by 100.
- Convert decimal to percent: multiply by 100.

Example

Convert the percentages to decimal numbers.

$$\begin{aligned} 40\% &= 40 \div 100 \\ &= 0.4 \end{aligned}$$

$$\begin{aligned} 83.1\% &= 83.1 \div 100 \\ &= 0.831 \end{aligned}$$

$$\begin{aligned} 275\% &= 275 \div 100 \\ &= 2.75 \end{aligned}$$

Convert the decimal numbers to percentages.

$$\begin{aligned} 0.7 &= 0.7 \cdot 100\% \\ &= 70\% \end{aligned}$$

$$\begin{aligned} 0.042 &= 0.042 \cdot 100\% \\ &= 4.2\% \end{aligned}$$

$$\begin{aligned} 4.2 &= 4.2 \cdot 100 \\ &= 420\% \end{aligned}$$

Fractions to Decimals

All rational numbers can be written as an integer, a terminating decimal, or a repeating decimal. We can do this by treating a fraction as division.

Example

Write each fraction in decimal form without using a calculator.

$$\begin{aligned} \frac{3}{5} &= \frac{6}{10} \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} \frac{11}{25} &= \frac{44}{100} \\ &= 0.44 \end{aligned}$$

$$\begin{aligned} 4\frac{3}{4} &= 4\frac{75}{100} \\ &= 4.75 \end{aligned}$$

Write each fraction in decimal form using a calculator.

$$\begin{aligned} \frac{97}{80} &= 97 \div 80 \\ &= 1.2125 \end{aligned}$$

$$\begin{aligned} \frac{8}{11} &= 8 \div 11 \\ &= 0.72727272\ldots \\ &= 0.\overline{72} \end{aligned}$$

$$\begin{aligned} \frac{49}{15} &= 49 \div 15 \\ &= 3.26666666\ldots \\ &= 3.2\overline{6} \end{aligned}$$

Decimals to Fractions

Each place value after the decimal point represents dividing by a larger power of ten.

$$0.1 = \frac{1}{10} \qquad 0.01 = \frac{1}{100} \qquad 0.001 = \frac{1}{1000} \qquad 0.0001 = \frac{1}{10000}$$

Any terminating decimal decimal can be written as a fraction. The number of digits after the decimal point tells us how many zeroes the denominator should have.

Example

Write each as a fraction.

$$\begin{aligned} 0.65 &= \frac{65}{100} \\ &= \frac{13}{20} \end{aligned} \qquad \begin{aligned} 3.4 &= 3\frac{4}{10} \\ &= 3\frac{2}{5} \end{aligned} \qquad \begin{aligned} 0.425 &= \frac{425}{1000} \\ &= \frac{17}{40} \end{aligned} \qquad \begin{aligned} 1.012 &= 1\frac{12}{1000} \\ &= 1\frac{3}{250} \end{aligned}$$

For repeating decimals, we can use the property that $0.\overline{9} = 0.99999... =$ 1.

$$0.\overline{1} = \frac{0.\overline{1}}{0.\overline{9}} = \frac{1}{9} \qquad 0.\overline{1} = \frac{0.\overline{01}}{0.\overline{99}} = \frac{1}{99} \qquad 0.\overline{1} = \frac{0.\overline{001}}{0.\overline{999}} = \frac{1}{999}$$

Example

Write each as a fraction.

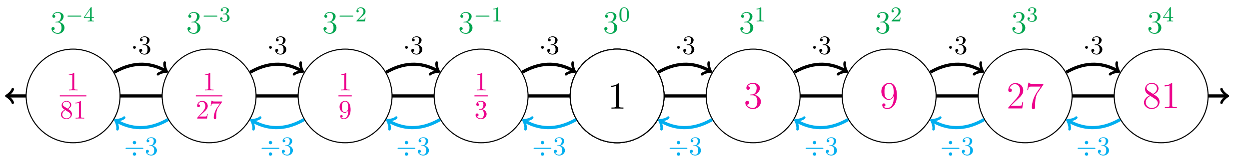
$$\begin{aligned} 0.\overline{6} &= \frac{6}{9} \\ &= \frac{2}{3} \end{aligned} \qquad \begin{aligned} 0.\overline{45} &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned} \qquad \begin{aligned} 0.\overline{259} &= \frac{259}{999} \\ &= \frac{7}{27} \end{aligned}$$

$$\begin{aligned} 0.7\overline{3} &= \frac{1}{10} \cdot 7.\overline{3} \\ &= \frac{1}{10} \cdot 7\frac{3}{9} \\ &= \frac{1}{10} \cdot 7\frac{1}{3} \\ &= \frac{1}{10} \cdot \frac{22}{3} \\ &= \frac{22}{30} \end{aligned} \qquad \begin{aligned} 0.11\overline{8} &= \frac{1}{100} \cdot 11.\overline{8} \\ &= \frac{1}{100} \cdot 11\frac{8}{9} \\ &= \frac{1}{100} \cdot \frac{107}{9} \\ &= \frac{107}{900} \end{aligned} \qquad \begin{aligned} 0.1\overline{28} &= \frac{1}{10} \cdot 1.\overline{28} \\ &= \frac{1}{10} \cdot 1\frac{28}{99} \\ &= \frac{1}{10} \cdot \frac{127}{99} \\ &= \frac{127}{990} \end{aligned}$$

2.1 Positive and Negative Exponents

An expression in the form a^m can be used to represent repeated multiplication. The base, a, is the value to be multiplied, and the exponent, m, is the number of a's being multiplied. We can read the expression as "a to the power of m".

Here are some of the powers when the base is 3:



Example

Write the expressions in expanded form, and then evaluate them.

$3^4 = 3 \cdot 3 \cdot 3 \cdot 3$
$= 81$

$4^3 = 4 \cdot 4 \cdot 4$
$= 64$

$11^2 = 11 \cdot 11$
$= 121$

Write the expressions in expanded form.

$x^6 = x \cdot x \cdot x \cdot x \cdot x \cdot x$

$y^5 = y \cdot y \cdot y \cdot y \cdot y$

$a^4 = a \cdot a \cdot a \cdot a$

Write the expressions in exponent form.

$7 \cdot 7 \cdot 7 = 7^3$

$12 \cdot 12 \cdot 12 \cdot 12 \cdot 12 = 12^5$

$x \cdot x = x^2$

If the exponent is negative, we need to repeat the opposite of multiplication, which is division. If the base is an integer, this usually results in a fraction.

Example

Write the expressions in expanded form, and then evaluate them.

$$3^{-2} = \frac{1}{3 \cdot 3} = \frac{1}{9}$$

$$2^{-5} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{32}$$

$$10^{-3} = \frac{1}{10 \cdot 10 \cdot 10} = \frac{1}{1000}$$

Write the expressions in expanded form.

$$x^{-4} = \frac{1}{x \cdot x \cdot x \cdot x}$$

$$y^{-2} = \frac{1}{y \cdot y}$$

$$b^{-7} = \frac{1}{b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b}$$

Write the expressions in exponent form.

$$\frac{1}{6 \cdot 6 \cdot 6} = 6^{-3}$$

$$\frac{1}{9 \cdot 9 \cdot 9 \cdot 9} = 9^{-4}$$

$$\frac{1}{y \cdot y \cdot y \cdot y \cdot y \cdot y} = y^{-6}$$

2.2 Exponent Rules with the Same Base

Example

Write these expressions in expanded form, then simplify as single exponents.

$$\begin{aligned} 3^5 \cdot 3^2 &= (3\cdot3\cdot3\cdot3\cdot3)\cdot(3\cdot3) \\ &= 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 \\ &= 3^7 \end{aligned}$$

$$\begin{aligned} \frac{5^9}{5^3} &= \frac{5\cdot5\cdot5\cdot5\cdot5\cdot5\cdot\cancel{5\cdot5\cdot5}}{\cancel{5\cdot5\cdot5}} \\ &= 5\cdot5\cdot5\cdot5\cdot5\cdot5 \\ &= 5^6 \end{aligned}$$

Rule 1: The Exponent Product Rule

$$a^m \cdot a^n = a^{m+n}$$

Multiplying expressions with the **same base** is equivalent to adding the exponents.

Rule 2: The Exponent Quotient Rule

$$\frac{a^m}{a^n} = a^{m-n}$$

Dividing expressions with the **same base** is equivalent to subtracting the exponents.

Example

Simplify each using the Exponent Product Rule.

$2^8 \cdot 2^3 = 2^{11}$ $7^6 \cdot 7^{13} = 7^{19}$ $x^5 \cdot x^9 = x^{14}$

Simplify each using the Exponent Quotient Rule.

$\frac{6^{14}}{6^5} = 6^9$ $\frac{4^3}{4^8} = 4^{-5}$ $\frac{t^{10}}{t^7} = t^3$

Example

Write these expressions in expanded form, then simplify using single **positive** exponents.

$$\begin{aligned} (2^3)^4 &= 2^3\cdot2^3\cdot2^3\cdot2^3 \\ &= (2\cdot2\cdot2)\cdot(2\cdot2\cdot2)\cdot(2\cdot2\cdot2)\cdot(2\cdot2\cdot2) \\ &= 2^{12} \end{aligned}$$

$$\begin{aligned} a^{-5} &= \frac{1}{a\cdot a\cdot a\cdot a\cdot a} \\ &= \frac{1}{a^5} \end{aligned}$$

Rule 3: The Exponent Power Rule

$$(a^m)^n = a^{mn}$$

Raising a base to a power then another is equivalent to multiplying the exponents.

Rule 4: The Negative Exponent Rule

$$a^{-m} = \frac{1}{a^m}$$

Changing the sign of an exponent is equivalent to taking the reciprocal of the expression.

Example

Simplify each using the Exponent Power Rule.

$(3^4)^2 = 3^8$

$(10^5)^3 = 10^{15}$

$(b^7)^6 = b^{42}$

Write using a positive exponent.

$5^{-7} = \frac{1}{5^7}$

Write without using a fraction.

$\frac{1}{e^{11}} = e^{-11}$

Special Exponent Values

$$a^0 = 1 \quad (a \neq 0)$$

Any exponential expression with zero for the exponent (and the base is not zero) is equal to one.

$$a^1 = a$$

Any exponential expression with one for the exponent is equal to the base.

Example

Simplify each expression with a positive exponent. State which rule is used in each step.

$\frac{t^8}{t^{11}} \cdot t^5 = t^{-3} \cdot t^5$ Rule 2

$= t^2$ Rule 1

$s^5 (s^4)^7 = s^5 \cdot s^{28}$ Rule 3

$= s^33$ Rule 1

$\frac{(a^2)^3}{a^{13}} = \frac{a^6}{a^{13}}$ Rule 3

$= a^{-7}$ Rule 2

$= \frac{1}{a^7}$ Rule 4

$\frac{b^{22}}{(b^2 \cdot b^4)^3} = \frac{b^{22}}{(b^6)^3}$ Rule 1

$= \frac{b^{22}}{b^{18}}$ Rule 3

$= b^4$ Rule 2

Example

Simplify each expression.

$a^3b^5 \cdot a^7b = a^{10}b^6$

$\frac{x^5y^2}{x^4y^8} = xy^{-6}$

$\frac{s^4t^5 \cdot s^2}{t^2} = s^6t^3$

2.3 Exponent Rules with the Same Exponent

Example

Write these expressions in expanded form, then simplify each using a single base.

$$\begin{aligned} 2^4 \cdot 3^4 &= (2 \cdot 2 \cdot 2 \cdot 2) \cdot (3 \cdot 3 \cdot 3 \cdot 3) \\ &= (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \\ &= 6 \cdot 6 \cdot 6 \cdot 6 \\ &= 6^4 \end{aligned}$$

$$\begin{aligned} \frac{12^5}{4^5} &= \frac{12 \cdot 12 \cdot 12 \cdot 12 \cdot 12}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} \\ &= \frac{12}{4} \cdot \frac{12}{4} \cdot \frac{12}{4} \cdot \frac{12}{4} \cdot \frac{12}{4} \\ &= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \\ &= 3^5 \end{aligned}$$

Rule 5: The Base Product Rule

$$a^m \cdot b^m = (ab)^m$$

Multiplying expressions with the **same exponent** is equivalent to multiplying the bases.

Rule 6: The Base Quotient Rule

$$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

Dividing expressions with the **same exponent** is equivalent to dividing the bases.

Example

Simplify each of the following. Write your answer as a single exponent.

$$\begin{aligned} 3^7 \cdot 5^7 &= (3 \cdot 5)^7 \\ &= 15^7 \end{aligned}$$

$$\begin{aligned} 2^4 \cdot 9^4 &= (2 \cdot 9)^4 \\ &= 18^4 \end{aligned}$$

$$\begin{aligned} \frac{63^5}{9^5} &= \left(\frac{63}{9}\right)^5 \\ &= 7^5 \end{aligned}$$

Simplify and evaluate each of the following.

$$\begin{aligned} \frac{(2^5 \cdot 3)^3}{2^{11} \cdot 3^2} &= \frac{(2^5)^3 \cdot 3^3}{2^{11} \cdot 3^2} \\ &= \frac{2^{15} \cdot 3^3}{2^{11} \cdot 3^2} \\ &= 2^4 \cdot 3 \\ &= 48 \end{aligned}$$

$$\begin{aligned} \frac{10^2 \cdot 10^4 \cdot 5}{5^7} &= \frac{10^6}{5^6} \\ &= \left(\frac{10}{5}\right)^6 \\ &= 2^6 \\ &= 64 \end{aligned}$$

Simplify each of the following. Don't use fractions for your final expressions.

$$\begin{aligned} \frac{(ab)^2}{b^5} &= \frac{a^2b^2}{b^5} \\ &= a^2b^{-3} \end{aligned}$$

$$\begin{aligned} \frac{(3x)^4}{x^5} &= \frac{3^4x^4}{x^5} \\ &= 81x^{-1} \end{aligned}$$

2.4 Scientific Notation

The decimal number system is base ten, which means each place value corresponds to a different power of ten.

- If n is positive, then 10^n is 1 shifted n place values to the left.
- If n is negative, then 10^n is 1 shifted $|n|$ place values to the right.

Example

Write in decimal notation:

$10^5 = 100\,000$ $10^{-4} = 0.0001$ $10^3 = 1\,000$

Write as an exponent of 10:

$0.000001 = 10^{-6}$ $10\,000\,000 = 10^7$ $0.01 = 10^{-2}$

Scientific notation is a way of writing numbers which uses leading digits multiplied by a power of ten. The leading digits always have a single non-zero digit before the decimal point, with the power of ten used to shift the place value.

Scientific notation with positive powers can represent very big numbers, and scientific notation with negative powers can represent very small numbers.

Example

Write in ordinary decimal notation:

$7.482 \times 10^5 = 748\,200$ $5.213 \times 10^{-4} = 0.0005213$ $3.9742 \times 10^3 = 3\,974.2$

Write in scientific notation:

$0.00000358 = 3.58 \times 10^{-6}$ $34\,910\,000 = 3.491 \times 10^7$ $0.0882 = 8.82 \times 10^{-2}$

These are not in valid scientific notation. Correct them.

$$\begin{aligned} 12.3 \times 10^8 &= 1.23 \times 10^1 \times 10^8 \\ &= 1.23 \times 10^9 \end{aligned}$$

$$\begin{aligned} 0.0234 \times 10^5 &= 2.34 \times 10^{-2} \times 10^5 \\ &= 2.34 \times 10^3 \end{aligned}$$

The exponent on the ten is sometimes called the order of magnitude. To compare two numbers in scientific notation, compare the order of magnitude first. If these are the same, the numbers have similar size, so we compare their leading digits.

Example

Which is larger of 7.452×10^{-6} and 3.529×10^{-2}?

3.529×10^{-2} is much larger, as the exponent is larger.

Compare the sizes of a bacterium with a diameter of 1.5×10^{-6} m, a virus with a diameter of 4.5×10^{-8} m, and a red blood cell with a diameter of 8.2×10^{-6} m.

The virus is much smaller than both the red blood cell and the bacterium. The bacterium is smaller than the red blood cell.

2.5 Operations in Scientific Notation

To multiply and divide numbers in scientific notation, the leading digits can be treated as ordinary numbers, and the exponents can be simplified using exponent rules. Always check that the answer is in correct scientific notation.

Example

Evaluate each of the following.

$$\begin{aligned}(3.5 \times 10^8)(5 \times 10^{-3}) &= (3.5 \times 5) \times (10^8 \times 10^{-3})\\ &= 17.5 \times 10^5\\ &= 1.75 \times 10^1 \times 10^5\\ &= 1.75 \times 10^6\end{aligned}$$

$$\begin{aligned}\frac{1.8 \times 10^{11}}{6 \times 10^7} &= 0.3 \times 10^4\\ &= 3 \times 10^{-1} \times 10^4\\ &= 3 \times 10^3\end{aligned}$$

$$\begin{aligned}(5 \times 10^{-4})(9 \times 10^{-9}) &= (5 \times 9) \times (10^{-4} \times 10^{-9})\\ &= 45 \times 10^{-13}\\ &= 4.5 \times 10^1 \times 10^{-13}\\ &= 4.5 \times 10^{-12}\end{aligned}$$

$$\begin{aligned}\frac{5.6 \times 10^5}{8 \times 10^{18}} &= 0.7 \times 10^{-13}\\ &= 7 \times 10^{-1} \times 10^{-13}\\ &= 7 \times 10^{-14}\end{aligned}$$

Example

The earth is 1.496×10^{11} m from the sun. Light travels at 3.0×10^8 m each second. How many seconds does it take light from the sun to reach the earth? *Use a calculator.*

$$\begin{aligned}\frac{1.496 \times 10^{11}\ \text{m}}{3.0 \times 10^8\ \text{m/s}} &= 4.99 \times 10^2\ s\\ &= 499\ \text{s}\end{aligned}$$

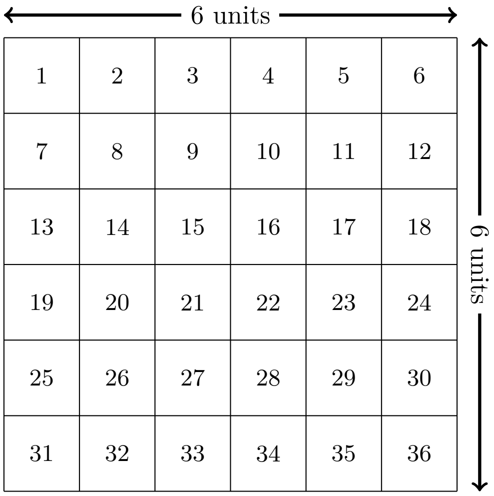
2.6 Square Roots

If we want to make a square whose sides are n units long, we'll need $n \cdot n = n^2$ unit squares. This is why multiplying a number by itself, or applying an exponent of two is called squaring.

Example

How many unit squares form a square with sides six units long?

$$6^2 = 6 \cdot 6 = 36$$



1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

The inverse (the opposite) operation of squaring is the square root, which is represented by the radical symbol $\sqrt{\ }$. The number underneath a radical is called the radicand.

$\sqrt{n}$ is the number whose square is equal to n.

Example

What is the side length of a square made from 36 unit squares?

$$\sqrt{36} = 6$$

A number which results from squaring a whole number is called a perfect square:

$1^2 = 1$

$2^2 = 4$

$3^2 = 9$

$4^2 = 16$

$5^2 = 25$

$6^2 = 36$

$7^2 = 49$

$8^2 = 64$

$9^2 = 81$

$10^2 = 100$

$11^2 = 121$

$12^2 = 144$

$13^2 = 169$

$14^2 = 196$

$15^2 = 225$

$16^2 = 256$

$17^2 = 289$

$18^2 = 324$

$19^2 = 361$

$20^2 = 400$

The square root of a perfect square is a whole number. The square root of any other whole number is between whole numbers. These square roots can only be approximated when using finite decimal places.

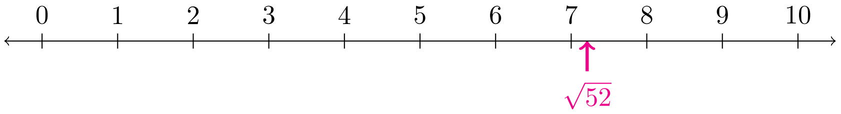
Example

Evaluate $\sqrt{289}$, and give a reason for your answer.

$\sqrt{289} = 17$, because $17^2 = 17 \cdot 17 = 289$.

Example

Approximately locate $\sqrt{52}$ on a number line. Explain why the estimate has this location.



52 is between $7^2 = 49$ and $8^2 = 64$, so $\sqrt{52}$ is between 7 and 8. We can expect $\sqrt{52}$ to be closer to 7 than to 8.

Approximate the value of $\sqrt{25}$ with a calculator.

$\sqrt{52} \approx 7.211$

2.7 Understanding Irrational Numbers

A set is a collection of mathematical items, which is often a collection of numbers.

- The whole numbers are the numbers used for counting, including zero.
- The integers are the whole numbers along with their negative counterparts.
- The rational numbers are the numbers which can be written as a fraction (or "ratio") with two integers.

Two new number sets to consider:

- The real numbers are the numbers which can be placed on the number line.
- The irrational numbers are the real numbers which are not rational.

Rational and Irrational Numbers

We've already seen that integers, terminating decimals and repeating decimals can all be written as fractions using integers, so they are always rational numbers. In fact, every rational number is one of these three.

Therefore, any other number must be an irrational number.

A decimal which doesn't terminate and doesn't repeat is irrational.

The square root of a whole number which is not a perfect square is irrational.

$\pi = 3.14159...$ is irrational.

Combining Rational and Irrational Numbers

The sum or product of two rational numbers is always rational.

Why this is true:

If two numbers are rational, that means they can be represented by fractions. Adding two fractions makes a fraction, and multiplying two fractions makes a fraction, so the sum or product is always rational.

Another way of describing this is to say that the rational numbers are closed under addition and multiplication. Just like you can't leave a room if it is closed, we can't leave the closed rational numbers by adding or multiplying.

The sum or product of two irrational numbers is sometimes irrational, but not always.

Example

Think of a pair of irrational numbers whose sum is rational.

$\sqrt{5}$ and $-\sqrt{5}$, because $\sqrt{5} + (-\sqrt{5}) = 0$ is a rational number.

Think of a pair of irrational numbers whose product is rational.

$\sqrt{3}$ and $\sqrt{12}$, because $\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = 6$ is a rational number.

This means the irrational numbers are not closed under addition or multiplication.

The sum of a rational and irrational number is always irrational.

The product of a (non-zero) rational number and an irrational number is always irrational.

Example

Answer true or false. Give a reason for each answer.

The product of a rational number and an irrational number is never irrational.

TRUE, because the sum is always rational.

$3 + \pi$ is a rational number.

FALSE, because 3 is rational and π is irrational, so their sum must be irrational.

$\frac{2}{3} \cdot \sqrt{25}$ is irrational, because it is a product of a non-zero rational number and a square root.

FALSE, $\sqrt{25} = 5$ is rational, because 25 is a perfect square. The product is rational.

3.1 The Order of Operations

A numerical expression is a combination of numbers and operations which represents a numerical value. To evaluate an expression means to determine that overall value. When evaluating expressions, we follow the order of operations.

G	Grouping Symbols including: (**in parentheses**), [**in brackets**], {**in braces**}, $\lvert\textbf{in absolute value bars}\rvert$, $\sqrt{\textbf{under a radical}}$, and $\dfrac{\textbf{numerator of a fraction}}{\textbf{denominator of a fraction}}$.
E	Exponents, which includes $\textbf{evaluating}^{\textbf{powers}}$ and $\sqrt{\textbf{evaluating radicals}}$.
$\overrightarrow{\textbf{M D}}$	Multiplication and Division, in order from **left-to-right**.
$\overrightarrow{\textbf{A S}}$	Addition and Subtraction, in order from **left-to-right**.

To show your working clearly, you should write your calculations one step at a time. We use the equals symbol to indicate that expressions as equivalent. You should always work vertically, with all the equals signs written in a straight line.

Example

Evaluate each expression.

$$\begin{aligned}3\,(8-3)^2-5\cdot 7 &= 3\cdot 5^2-5\cdot 7\\ &= 3\cdot 25-5\cdot 7\\ &= 75-35\\ &= 40\end{aligned}$$

$$\begin{aligned}\frac{4-3(-6)}{5(-3)+17} &= \frac{4-(-18)}{-15+17}\\ &= \frac{22}{2}\\ &= 11\end{aligned}$$

Evaluating Exponents

Example

Write each expression in expanded form, and then evaluate.

$$\begin{aligned}(-2)^3 &= (-2)\,(-2)\,(-2)\\ &= -8\end{aligned}$$

$$\begin{aligned}(-2)^4 &= (-2)\,(-2)\,(-2)\,(-2)\\ &= 16\end{aligned}$$

$$\begin{aligned}-2^3 &= -2\cdot 2\cdot 2\\ &= -8\end{aligned}$$

$$\begin{aligned}-2^4 &= -2\cdot 2\cdot 2\cdot 2\\ &= -16\end{aligned}$$

- A negative base to an odd power is always negative.
- A negative base to an even power is always positive.
- A negative sign not contained in grouping symbols with the base is not part of the base, and will be evaluated after the exponent.

Example

Evaluate each of the expressions.

$$(-3)^4 + (-4)^3 = 81 + (-64)$$
$$= 17$$

$$(-3)^2 + (-3)^3 - 3^4 = 9 + (-27) - 81$$
$$= -18 - 81$$
$$= -99$$

Expressions Represented with Words

related to +	related to −	related to ×	related to ÷
plus	minus	times	divide
sum	difference	product	quotient
addition	subtraction	multiplication	division
more than	less than	twice, double, triple	half of, third of
increased by	decreased by	of	split evenly

Example

Write each description as a numerical expression, then evaluate.

The quotient of 20 and 4.

$$\frac{20}{4} = 5$$

25 less than 8.

$$8 - 25 = -17$$

Twice the difference of 13 and 9.

$$2(13 - 9) = 2 \cdot 4$$
$$= 8$$

10 more than the product of 9 and 7.

$$9 \cdot 7 + 10 = 63 + 10$$
$$= 73$$

Half of the sum of 14 and 8.

$$\frac{14 + 8}{2} = \frac{22}{2}$$
$$= 11$$

The sum of 14 and half of 8

$$14 + \frac{8}{2} = 14 + 4$$
$$= 18$$

7 subtracted from the square root of 16.

$$\sqrt{16} - 7 = 4 - 7$$
$$= -3$$

The square of the quantity 18 minus 7.

$$(18 - 7)^2 = 11^2$$
$$= 121$$

3.2 Variables and Substitution

A variable is a quantity whose value we don't know yet or whose value can change. A variable is usually represented by a letter.

An algebraic expression is an expression which contains variables as well as numbers and operations.

If we know the values of the variables, we can substitute the variables by replacing them with their values. This turns an algebraic expression into a numerical expression, which can be evaluated. Always surround values with parentheses when substituting.

Example

Suppose that $a = 5$, $b = -7$, and $c = 2$. Evaluate each expression using these values.

$$\begin{aligned} 2a + 3b &= 2(5) + 3(-7) \\ &= 10 - 21 \\ &= -11 \end{aligned}$$

$$\begin{aligned} \sqrt{b^2 - 4ac} &= \sqrt{(-7)^2 - 4(5)(2)} \\ &= \sqrt{49 - 40} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

Example

Write each as an algebraic expression, where value of "a number" is represented by n.

Triple the sum of a number and 5.

$$3(n+5)$$

10 less than the square of a number.

$$n^2 - 10$$

Evaluate each expression where the value of "a number" is 2. $n = 2$

$$\begin{aligned} 3\,(n+5) &= 3\,((2)+5) \\ &= 3\,(7) \\ &= 21 \end{aligned}$$

$$\begin{aligned} n^2 - 10 &= (2)^2 - 10 \\ &= 4 - 10 \\ &= -6 \end{aligned}$$

Evaluate each expression where the value of "a number" is -8. $n = -8$

$$\begin{aligned} 3\,(n+5) &= 3\,((-8)+5) \\ &= 3\,(-3) \\ &= -9 \end{aligned}$$

$$\begin{aligned} n^2 - 10 &= (-8)^2 - 10 \\ &= 64 - 10 \\ &= 54 \end{aligned}$$

Example

Penelope's Perfect Pizza sells large pizzas for \$6 each, and also charges \$8 for delivery.

Choose a variable to represent the number of pizzas delivered to a customer.

Let p be the number of pizzas delivered to a customer.

Write an expression representing the total cost to a customer.

The cost to a customer is $6p + 8$.

Use your expression to find the cost to a customer who orders 4 pizzas.

$$p = 4 \qquad \Longrightarrow \qquad \begin{aligned} 6p + 8 &= 6\,(4) + 8 \\ &= 24 + 8 \\ &= \$32 \end{aligned}$$

Parts of an Algebraic Expression

Terms are the parts of an expression separated by plus and minus symbols. A term is often written as a product of a number and variables, sometimes with exponents.

The coefficient of a term is the number which multiplies the variables in the term. The sign of the coefficient is determined by the operation before the term.

A constant term is a term which doesn't contain any variables.

Example

List the terms of the expression $2x^2 + 3xy - 7y^2 + x - 9y + 14$.

The terms are $2x^2$, $3xy$, $-7y^2$, $-9y$ and 14.

What are the coefficients of the terms?

The coefficient of x^2 is 2

The coefficient of y^2 is -7

The coefficient of xy is 3

The coefficient of x is 1

The coefficient of y is -9

What is the constant term?

The constant term is 14.

3.3 Combining Like Terms

Two expressions are equivalent if their values are the same as each other for any values of their variables.

Example

Complete the tables by evaluating the expressions.

x	$7x$	$2x$	$7x+2x$	$9x$
-3	-21	-6	-27	-27
-1	-7	-2	-9	-9
2	14	4	18	18
5	35	10	45	45

What do you notice?

What do you wonder?

x	$3x$	$3x+8$	$11x$
-2	-6	2	-22
1	3	11	11
4	12	20	44
6	18	26	66

What do you notice?

What do you wonder?

x	y	$6x$	$4y$	$6x+4y$	$10xy$
-2	3	-12	12	0	-60
1	5	6	20	26	50
4	-1	24	-4	20	-40
6	7	36	28	64	420

What do you notice?

What do you wonder?

Like terms are two or more terms whose combinations of variables are equivalent.

Constant terms are also considered to be like terms with each other.

Expressions with like terms can be simplified by combining like terms into an equivalent single term by adding the coefficients.

Example

Does $7x + 2x$ have like terms? Does $3x + 8$ have like terms? Does $6x + 4y$ have like terms?

Only $7x + 2x$ has like terms, because both terms have the variable x.

$$7x + 2x = 9x$$

Example

If these are like terms, simplify them. If they are not, explain why.

$6a + 10a = 16a$ Like terms. Both have a.

$4s - 9t$ Not like terms. s and t are not the same variable.

$5y^2 - 12y^2 = -7y^2$ Like terms. Both have y^2.

$-2n^2 + 5n$ Not like terms. n^2 and n are not equivalent.

$-3 + 8 = 5$ Like terms. Constant terms are like with each other.

Example

Simplify $4x + 5x - 8y + 6y + 7 - 3$.

$$\underbrace{4x + 5x}_{\text{like terms}} \underbrace{-8y + 6y}_{\text{like terms}} \underbrace{+7 - 3}_{\text{like terms}} = 9x - 2y + 4$$

COMMUTATIVE PROPERTY OF ADDITION

Sums with swapped **terms**

are equivalent **.**

$$a + b = b + a$$

COMMUTATIVE PROPERTY OF MULTIPLICATION

Products with swapped **factors**

are equivalent **.**

$$ab = ba$$

Example

Simplify each of the following expressions by combining like terms.

$$\begin{aligned} 5s + 4t - 8s + 6t &= 5s - 8s + 4t + 6t \\ &= -3s + 10t \end{aligned}$$

$$\begin{aligned} 4x - 15x - 9 + 7x &= 4x - 15x + 7x - 9 \\ &= -4x - 9 \end{aligned}$$

$$\begin{aligned} 9cd - 2dc &= 9cd - 2cd \\ &= 7cd \end{aligned}$$

$$\begin{aligned} 7ab - 6a + 3b + 5ba &= 7ab + 5ab - 6a + 3b \\ &= 12ab - 6a + 3b \end{aligned}$$

$$\begin{aligned} 3x^2y + 2yx^2 + 9xy^2 &= 3x^2y + 2x^2y + 9xy^2 \\ &= 5x^2y + 9xy^2 \end{aligned}$$

$$\begin{aligned} 5x + 7x^2 - x + x^2 &= 7x^2 + 1x^2 + 5x - 1x \\ &= 8x^2 + 4x \end{aligned}$$

3.4 The Distributive Property

Example

Complete the table by evaluating the expressions.

x	$x+4$	$3(x+4)$	$3x$	$3x+12$
-3	1	3	-9	3
1	5	15	3	15
5	9	27	15	27
10	14	42	30	42

What do you notice?

What do you wonder?

THE DISTRIBUTIVE PROPERTY

Multiplying a sum by a value is equivalent **to multiplying each term of the sum by that value before adding.**

$$a(b+c) = ab + ac$$

	b	$+c$
a	ab	$+ac$

The process of applying the distributive property is called distributing. The box method helps us to make sure that each term inside the parentheses is multiplied by the value outside the parentheses.

Example

Distribute each of the expressions.

$5(x+9) = 5x + 45$

	x	$+9$
5	$5x$	$+45$

$-2(y-7) = -2y + 14$

	y	-7
-2	$-2y$	$+14$

$7(2n-3) = 14n - 21$

	$2n$	-3
7	$14n$	-21

$t(t+7) = t^2 + 7t$

	t	$+7$
t	t^2	$+7t$

$-3p(q+5) = -3pq - 15p$

	q	$+5$
$-3p$	$-3pq$	$-15p$

$2u(3u-5) = 6u^2 - 10u$

	$3u$	-5
$2u$	$6u^2$	$-10u$

$-4(3a-5b-9) = -12a + 20b + 36$

	$3a$	$-5b$	-9
-4	$-12a$	$+20b$	$+36$

$2x(x+3y-5) = 2x^2 + 6xy - 10x$

	x	$+3y$	-5
$2x$	$2x^2$	$+6xy$	$-10x$

3.5 Factoring

Factoring is the opposite process of distributing. One way to do this is to find the greatest common factor, or GCF.

The first factor to find is the greatest common factor of all the coefficients.

Example

Factor the following expressions.

$7n - 21 = 7(n-3)$

	n	-3
7	$7n$	-21

$10x + 16 = 2(5x+8)$

	$5x$	$+8$
2	$10x$	$+16$

$15m - 50 = 5(3m-10)$

	$3m$	-10
5	$15m$	-50

$6a - 30 = 6(a-5)$

	a	-5
6	$6a$	-30

$28x + 70 = 14(2x+5)$

	$2x$	$+5$
14	$28x$	$+70$

$105t + 45 = 15(7t+3)$

	$7t$	$+3$
15	$105t$	$+45$

If all the terms share any variables in common, these are also factors of the GCF.

Example

Factor the following expressions.

$x^2 + 8x = x(x+8)$

	x	$+8$
x	x^2	$+8x$

$y^2 - 12y = y(y-12)$

	y	-12
y	y^2	$-12y$

$2a^2 - 14a = 2a(a-7)$

	a	-7
$2a$	$2a^2$	$-14a$

$8st + 4t = 4t(2s+1)$

	$2s$	$+1$
$4t$	$8st$	$+4t$

$12x^3 + 15x^2 = 3x^2(4x+5)$

	$4x$	$+5$
$3x^2$	$12x^3$	$+15x^2$

$4a^2b - 7ab = ab(4a-7)$

	$4a$	-7
ab	$4a^2b$	$-7ab$

3.6 Algebraic Reasoning

Much of what we do in algebra is based on the following algebraic properties.

associative property of addition	$(a+b)+c=a+(b+c)$	if we add three numbers, we can do either addition first
associative property of multiplication	$(a\cdot b)\cdot c=a\cdot(b\cdot c)$	if we multiply three numbers, we can do either multiplication first
commutative property of addition	$a+b=b+a$	we can change the order of terms in addition
commutative property of multiplication	$a\cdot b=b\cdot a$	we can change the order of factors in multiplication
distributive property	$a(b+c)=ab+ac$	we can distribute and factor

Most of the time we don't need to think about these properties to work algebraically. Sometimes, however, we need to prove our work by justifying our reasoning, using the properties.

We can also use operations as reasons for our calculations.

Example

Justify the following simplification, giving a reason for each step.

$3(x-4)+2(5x+7)=3(x)+3(-4)+2(5x)+2(7)$	distributive property
$=3(x)+3(-4)+(2\cdot 5)x+2(7)$	associative property of multiplication
$=3x+(-12)+10x+14$	multiplication
$=3x+10x+(-12)+14$	commutative property of addition
$=(3+10)\cdot x+(-12)+14$	distributive property
$=(3+10)\cdot x+(-12+14)$	associative property of addition
$=13x+2$	addition

4.1 Solving Equations

An equation is a mathematical statement which says that two expressions are equal. If the equation contains a variable, the value of that variable which makes the equation true (makes the two sides equal) is called a solution.

Example

Consider the equation $\frac{3x+6}{5} = -3$.

Show that $x = -7$ is a solution.

$$\begin{aligned}\frac{3(-7)+6}{5} &= \frac{-21+6}{5}\\ &= \frac{-15}{5}\\ &= -3\end{aligned}$$

$x = -7$ is a solution.

Show that $x = 8$ is **not** a solution.

$$\begin{aligned}\frac{3(8)+6}{5} &= \frac{24+6}{5}\\ &= \frac{30}{5}\\ &= 6\end{aligned}$$

$x = 8$ is not a solution.

Solving an equation means to find a solution for it.

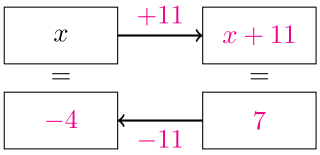
Solving Method 1: Backtracking

The backtracking method identifies the operations applied to the variable, then uses inverse operations to work back to the solution.

Example

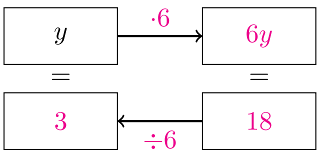
Solve each equation using the backtracking diagram.

$x + 11 = 7$



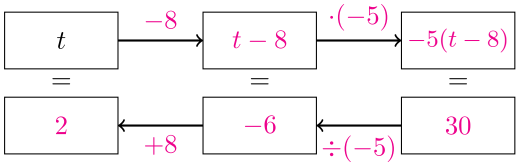
$$\begin{aligned}x + 11 &= 7\\ x &= -4\end{aligned}$$

$6y = 18$



$$\begin{aligned}6y &= 18\\ y &= 3\end{aligned}$$

$-5(t-8) = 30$



$$\begin{aligned}-5(t-8) &= 30\\ t-8 &= -6\\ t &= 2\end{aligned}$$

Example (continued)

$\frac{n}{4} + 11 = 5$

$$\begin{tikzcd}
n \arrow[r, "\div 4"] \arrow[d, phantom, "="] & \frac{n}{4} \arrow[r, "+11"] \arrow[d, phantom, "="] & \frac{n}{4}+11 \arrow[d, phantom, "="] \\
-24 & -6 \arrow[l, "\cdot 4"'] & 5 \arrow[l, "-11"']
\end{tikzcd}$$

$$\frac{n}{4} + 11 = 5$$
$$\frac{n}{4} = -6$$
$$n = -24$$

$\frac{3x-9}{2} = 6$

$$\begin{tikzcd}
x \arrow[r, "\times 3"] \arrow[d, phantom, "="] & 3x \arrow[r, "-9"] \arrow[d, phantom, "="] & 3x-9 \arrow[r, "\div 2"] \arrow[d, phantom, "="] & \frac{3x-9}{2} \arrow[d, phantom, "="] \\
7 & 21 \arrow[l, "\div 3"'] & 12 \arrow[l, "+9"'] & 6 \arrow[l, "\times 2"']
\end{tikzcd}$$

$$\frac{3x-9}{2} = 6$$
$$3x - 9 = 12$$
$$3x = 21$$
$$x = 7$$

The Properties of Equality

addition property of equality	$a = b$ if and only if $a + c = b + c$
subtraction property of equality	$a = b$ if and only if $a - c = b - c$
multiplication property of equality	$a = b$ if and only if $a \cdot c = b \cdot c$ (if $c \neq 0$)
division property of equality	$a = b$ if and only if $\frac{a}{c} = \frac{b}{c}$ (if $c \neq 0$)

Solving Method 2: Balancing Each Side

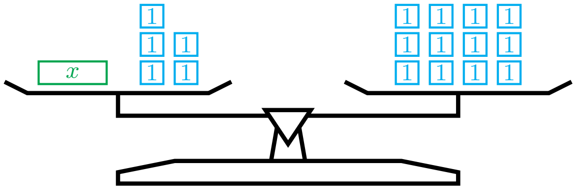
We can imagine an equation as a scale whose two sides perfectly balance. The scale remains balanced as long as we always do the same to both sides.

Example

Use the balance scales to illustrate each equation as you solve them.

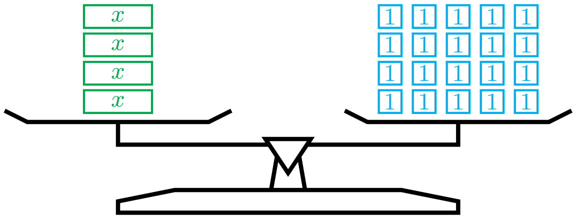
$x + 5 = 12$

$$x + 5 = 12$$
$$x + 5 - 5 = 12 - 5$$
$$x = 7$$



$4x = 20$

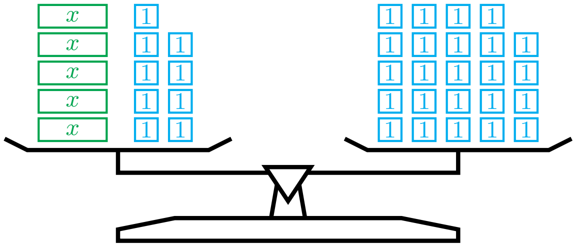
$$4x = 20$$
$$\frac{4x}{4} = \frac{20}{4}$$
$$x = 5$$



Example (continued)

$5x + 9 = 24$

$$5x + 9 = 24$$
$$5x + 9 - 9 = 24 - 9$$
$$5x = 15$$
$$\frac{5x}{5} = \frac{15}{5}$$
$$x = 3$$



Example

Solve each equation.

$n - 17 = -3$

$$n - 17 = -3$$
$$n - 17 + 17 = -3 + 17$$
$$n = 14$$

$\frac{b}{7} = 9$

$$\tfrac{b}{7} = 9$$
$$\tfrac{b}{7} \cdot 7 = 9 \cdot 7$$
$$b = 63$$

$-3t = -39$

$$-3t = -39$$
$$\frac{-3t}{-3} = \tfrac{-39}{-3}$$
$$t = 13$$

$2u - 9 = 15$

$$2u - 9 = 15$$
$$2u - 9 + 9 = 15 + 9$$
$$2u = 24$$
$$\tfrac{2u}{2} = \tfrac{24}{2}$$
$$u = 12$$

$\frac{x+15}{4} = 3$

$$\tfrac{x+15}{4} = 3$$
$$\tfrac{x+15}{4} \cdot 4 = 3 \cdot 4$$
$$x + 15 = 12$$
$$x + 15 - 15 = 12 - 15$$
$$x = -3$$

$2(y + 5) - 7 = 27$

$$2(y + 5) - 7 = 27$$
$$2(y + 5) - 7 + 7 = 27 + 7$$
$$2(y + 5) = 34$$
$$\tfrac{2(y+5)}{2} = \tfrac{34}{2}$$
$$y + 5 = 17$$
$$y + 5 - 5 = 17 - 5$$
$$y = 12$$

Example

Jessica is a member of a gym that charges $45 for membership, and an extra $6 for each visit. Jessica has paid $87 in total to the gym. How many visits has Jessica made to the gym?

Choose and define the variable.

Let v be the number of visits Jessica made to the gym.

Write the problem as an equation.

$$45 + 6v = 87$$

Solve the equation.

$$45 + 6v - 45 = 87 - 45$$
$$6v = 42$$
$$\tfrac{6v}{6} = \tfrac{42}{6}$$
$$v = 7$$

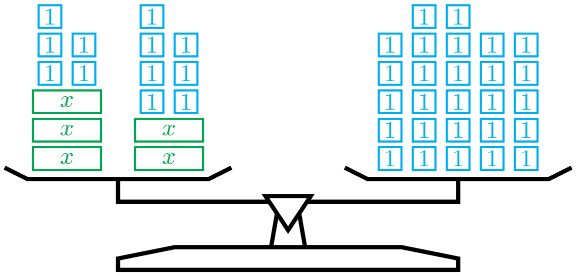
Jessica made 7 visits to the gym.

4.2 Equations with Simplifying

Example

Use the scale to illustrate $3x + 5 + 2x + 7 = 27$, and solve it.

$$3x + 5 + 2x + 7 = 27$$
$$5x + 12 = 27$$
$$5x + 12 - 12 = 27 - 12$$
$$5x = 15$$
$$\frac{5x}{5} = \frac{15}{5}$$
$$x = 3$$



Solve $6t - 9 - 8t + 21 = 2$.

$$6t - 9 - 8t + 21 = 2$$
$$-2t + 12 = 2$$
$$-2t + 12 - 12 = 2 - 12$$
$$-2t = -10$$
$$\frac{-2t}{-2} = \frac{-10}{-2}$$
$$t = 5$$

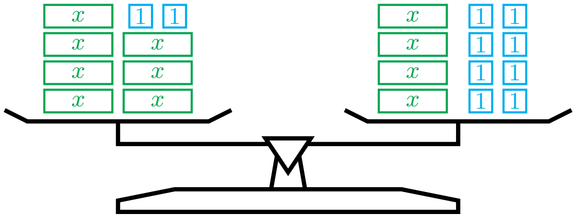
Solve $7 - 8n + 5n + 12n = 65 + 32$.

$$7 - 8n + 5n + 12n = 65 + 32$$
$$9n + 7 = 97$$
$$9n + 7 - 7 = 97 - 7$$
$$9n = 90$$
$$\frac{9n}{9} = \frac{90}{9}$$
$$n = 10$$

Example

Use the scale to illustrate $7x + 2 = 4x + 8$, and solve it.

$$7x + 2 = 4x + 8$$
$$7x - 4x + 2 = 4x - 4x + 8$$
$$3x + 2 = 8$$
$$3x + 2 - 2 = 8 - 2$$
$$3x = 6$$
$$\frac{3x}{3} = \frac{6}{3}$$
$$x = 2$$



Solve $5a = 56 - 2a$.

$$5a = 56 - 2a$$
$$5a + 2a = 56 - 2a + 2a$$
$$7a = 56$$
$$\frac{7a}{7} = \frac{56}{7}$$
$$a = 8$$

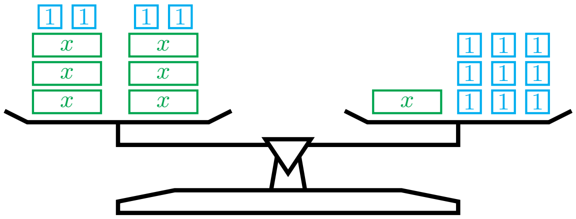
Solve $17 - b = 35 + 2b$.

$$17 - b = 35 + 2b$$
$$17 - b + b = 35 + 2b + b$$
$$17 = 35 + 3b$$
$$17 - 35 = 35 - 35 + 3b$$
$$-18 = 3b$$
$$\frac{-18}{3} = \frac{3b}{3}$$
$$b = -6$$

Example

Use the scale to illustrate $2(3x+2) = x+9$, and solve it.

$$\begin{aligned} 2(3x+2) &= x+9 \\ 6x+4 &= x+9 \\ 6x-x+4 &= x-x+9 \\ 5x+4 &= 9 \\ 5x+4-4 &= 9-4 \\ 5x &= 5 \\ \tfrac{5x}{5} &= \tfrac{5}{5} \\ x &= 1 \end{aligned}$$



Solve $9k+16-6(k+8) = 10$.

$$\begin{aligned} 9k+16-6(k+8) &= 10 \\ 9k+16-6k-48 &= 10 \\ 3k-32 &= 10 \\ 3k-32+32 &= 10+32 \\ 3k &= 42 \\ \tfrac{3k}{3} &= \tfrac{42}{3} \\ k &= 14 \end{aligned}$$

Solve $2(y-5)+3(4y+7) = -17$.

$$\begin{aligned} 2(y-5)+3(4y+7) &= -17 \\ 2y-10+12y+21 &= -17 \\ 14y+11 &= -17 \\ 14y+11-11 &= -17-11 \\ 14y &= -28 \\ \tfrac{14y}{14} &= \tfrac{-28}{14} \\ y &= -2 \end{aligned}$$

Solve $3\,(w+2) = 2\,(w-5)$.

$$\begin{aligned} 3(w+2) &= 2(w-5) \\ 3w+6 &= 2w-10 \\ 3w-2w+6 &= 2w-2w-10 \\ w+6 &= -10 \\ w+6-6 &= -10-6 \\ w &= -16 \end{aligned}$$

Solve $7(z-9) = -5(z+3)$.

$$\begin{aligned} 7(z-9) &= -5(z+3) \\ 7z-63 &= -5z-15 \\ 7z+5z-63 &= -5z+5z-15 \\ 12z-63 &= -15 \\ 12z-63+63 &= -15+63 \\ 12z &= 48 \\ \frac{12z}{12} &= \frac{48}{12} \\ z &= 4 \end{aligned}$$

1. If there are any **parentheses**, distribute them.
2. If the variable is on **both sides**, remove the term from one side by adding or subtracting.
3. If the variable is **repeated on one side**, simplify by combining like terms.
4. Finish solving as using inverse operations.

Example

Jayden starts jogging at a speed of 2 meters per second. Hailey waits 90 seconds, then starts jogging at a speed of 2.5 meters per second. How long will it take for Hailey to pass Jayden?

Choose and define the variable.

Let t be the number of seconds since Hailey started running.

Write the problem as an equation.

Jayen's distance $= 2(t + 90)$

Hailey's distance $= 2.5t$

$$2.5t = 2(t + 90)$$

Solve the equation.

$$\begin{aligned} 2.5t &= 2(t+90) \\ 2.5t &= 2t + 180 \\ 2.5t - 2t &= 2t - 2t + 180 \\ 0.5t &= 180 \\ \frac{0.5t}{0.5} &= \frac{180}{0.5} \\ t &= 360 \end{aligned}$$

Hailey will pass Jayden after 6 minutes.

4.3 Equations with Fractions

Approach 1: Solve while keeping fractions

When solving equations with fractions, we can still simplify them and use inverse operations to solve them as we would for equations with integers only.

Example

Solve $\frac{2a}{3} + \frac{5}{6} = \frac{4}{3}$.

$$\begin{aligned} \tfrac{2}{3}a + \tfrac{5}{6} &= \tfrac{4}{3} \\ \tfrac{2}{3}a + \tfrac{5}{6} - \tfrac{5}{6} &= \tfrac{4}{3} - \tfrac{5}{6} \\ \tfrac{2}{3}a &= \tfrac{8}{6} - \tfrac{5}{6} \\ &= \tfrac{3}{6} \\ &= \tfrac{1}{2} \\ \tfrac{3}{2} \cdot \tfrac{2}{3}a &= \tfrac{3}{2} \cdot \tfrac{1}{2} \\ a &= \tfrac{3}{4} \end{aligned}$$

Solve $\frac{2t+11}{4} + \frac{5t}{8} = \frac{16}{5}$.

$$\begin{aligned} \tfrac{1}{2}t + \tfrac{11}{4} + \tfrac{5}{8}t &= \tfrac{16}{5} \\ \tfrac{1}{2} + \tfrac{5}{8} &= \tfrac{4}{8} + \tfrac{5}{8} \\ &= \tfrac{9}{8} \\ \tfrac{9}{8}t + \tfrac{11}{4} &= \tfrac{16}{5} \\ \tfrac{9}{8}t + \tfrac{11}{4} - \tfrac{11}{4} &= \tfrac{16}{5} - \tfrac{11}{4} \\ &= \tfrac{64}{20} - \tfrac{55}{20} \\ \tfrac{9}{8}t &= \tfrac{9}{20} \\ \tfrac{8}{9} \cdot \tfrac{9}{8}t &= \tfrac{8}{9} \cdot \tfrac{9}{20} \\ t &= \tfrac{8}{20} \\ &= \tfrac{2}{5} \end{aligned}$$

Approach 2: Eliminate denominators first

Example

For each list of fractions, find the lowest common denominator.

$\frac{1}{2}, \frac{3}{4}, \frac{5}{8}$	$\frac{2}{3}, \frac{1}{5}, \frac{7}{10}$	$\frac{5}{4}, \frac{1}{6}, \frac{11}{12}$
LCD $= 8$	LCD $= 30$	LCD $= 12$

Multiply each fraction by the lowest common denominator, and simplify.

$\frac{1}{2} \cdot 8 = 4$	$\frac{2}{3} \cdot 30 = 2 \cdot 10 = 20$	$\frac{5}{4} \cdot 12 = 5 \cdot 3 = 15$
$\frac{3}{4} \cdot 8 = 3 \cdot 2 = 6$	$\frac{1}{5} \cdot 30 = 6$	$\frac{1}{6} \cdot 12 = 2$
$\frac{5}{8} \cdot 8 = 5$	$\frac{7}{10} \cdot 30 = 7 \cdot 3 = 21$	$\frac{11}{12} \cdot 12 = 11$

What do you notice?

What do you wonder?

The denominator of a fraction can be eliminated by multiplying the fraction by a multiple of the denominator. The LCD is a multiple of all the denominators in a set of fractions. This means we can eliminate all denominators in an equation by multiplying both sides by the LCD of all the fractions in the equation.

Example

Eliminate the denominators first before solving the equations.

Solve $\frac{2a}{3} + \frac{5}{6} = \frac{4}{3}$.

$$\begin{aligned} 6 \cdot \tfrac{2}{3}a + 6 \cdot \tfrac{5}{6} &= 6 \cdot \tfrac{4}{3} \\ 4a + 5 &= 8 \\ 4a + 5 - 5 &= 8 - 5 \\ 4a &= 3 \\ \tfrac{4a}{4} &= \tfrac{3}{4} \\ a &= \tfrac{3}{4} \end{aligned}$$

Solve $\frac{2t+11}{4} + \frac{5t}{8} = \frac{16}{5}$.

$$\begin{aligned} 40 \cdot \tfrac{2t+11}{4} + 40 \cdot \tfrac{5t}{8} &= 40 \cdot \tfrac{16}{5} \\ 10(2t+11) + 5 \cdot 5t &= 8 \cdot 16 \\ 20t + 110 + 25t &= 64 \\ 45t + 110 &= 128 \\ 45t + 110 - 110 &= 128 - 110 \\ 45t &= 18 \\ \tfrac{45t}{45} &= \tfrac{18}{45} \\ t &= \tfrac{2}{5} \end{aligned}$$

Which of the two approaches did you prefer? Why?

4.4 Number of Solutions

A solution to an equation is a value for the variable which makes the equation true. Many equations have exactly one solution, but this is not always the case.

Example

Use the table analyze the equation $3x + 5 = 3x + 7$.

x	LHS $3x + 5$	RHS $3x + 7$	Solution? LHS $\overset{?}{=}$ RHS
-2	-1	1	no
1	8	10	no
4	17	19	no
9	32	34	no

What do you notice?

What do you wonder?

Use the table analyze the equation $2(x - 3) = 2x - 6$.

x	LHS $2(x-3)$	RHS $2x - 6$	Solution? LHS $\overset{?}{=}$ RHS
-2	-10	-10	yes
1	-4	-4	yes
4	2	2	yes
9	12	12	yes

What do you notice?

What do you wonder?

If the two sides of an equation differ by a constant, then no number is a solution.

If the two sides of an equation are equivalent, then every number is a solution.

NUMBER OF SOLUTIONS
for linear equations with both sides distributed and simplified

variable term	constant term	type of solution
same coefficient	different constants	no solution
same coefficient	same constant	infinitely many solutions
different coefficients	N/A	one (unique) solution

Example

Determine the number of solutions each equation has. Justify your answers.

$3(2x+4)-2x+8=4(x+5)$

$$6x+12-2x+8=4x+20$$
$$4x+20=4x+20$$

The equation has infinitely many solutions because it has the same x coefficient and the same constant term on each side.

$4x+3-2(x-1)=5x+8$

$$4x+3-2x+2=5x+8$$
$$2x+5=5x+8$$
$$\ldots\, x=-1$$

The equation has one solution because it has different x coefficients on each side.

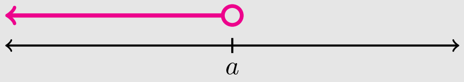
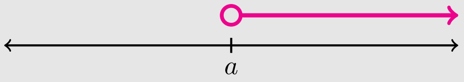
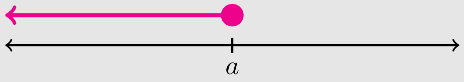
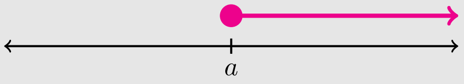
$2(5x-3)+4x=7(2x-1)$

$$10x-6+4x=14x-7$$
$$14x-6=14x-7$$

The equation has no solutions because it has the same x coefficient but different constant terms on each side.

4.5 Linear Inequalities

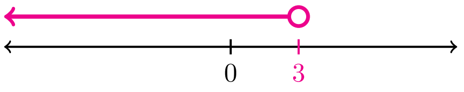
An inequality is a statement similar to an equation, but doesn't use equals.

x **is less than (not equal to)** a	$x < a$	
x **is greater than (not equal to)** a	$x > a$	
x **is less than or equal to** a	$x \leq a$	
x **is greater than or equal to** a	$x \geq a$	

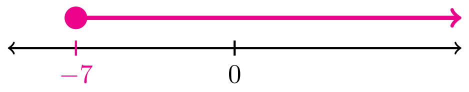
Example

Write each description as an inequality, and plot it on the number line.

x is below 3. $x < 3$



x is at least -7. $x \geq -7$

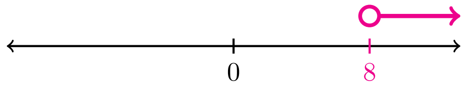


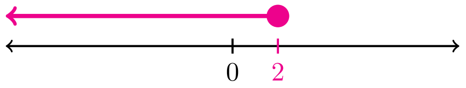
Example

Use the tables analyze the inequalities $2x - 7 > 9$ and $-3t + 5 \geq -1$. Then plot the solution on the number line.

x	$2x - 7$	$2x - 7 \overset{?}{>} 10$
5	3	no
8	9	no (boundary)
10	13	yes
12	37	yes

t	$-3t + 5$	$-3t + 5 \overset{?}{\geq} -1$
-1	8	yes
1	2	yes
2	-1	yes (boundary)
5	-10	no





What do you notice?

What do you wonder?

Solve the inequalities algebraically.

$$\begin{gathered} 2x - 7 > 9 \\ 2x - 7 + 7 > 9 + 7 \\ 2x > 16 \\ \frac{2x}{2} > \frac{16}{2} \\ x > 8 \end{gathered}$$

$$\begin{gathered} -3x + 5 \geq -1 \\ -3x + 5 - 5 \geq -1 - 5 \\ -3x \geq -6 \\ \frac{-3x}{-3} \geq \frac{-6}{-3} \\ x \leq 2 \end{gathered}$$

When solving inequalities, apply the same operation to both sides **.**

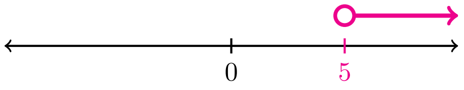
To add or subtract	To multiply or divide a positive	To multiply or divide a negative
keep the inequality	keep the inequality	reverse the inequality

Example

Solve each inequality, and use a number line to represent the solution set.

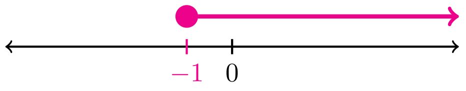
$3y - 5 > 10$.

$$\begin{aligned} 3y - 5 &> 10 \\ 3y - 5 + 5 &> 10 + 5 \\ 3y &> 15 \\ \frac{3y}{3} &> \frac{15}{3} \\ y &> 5 \end{aligned}$$



$-8u + 12 \leq -4$.

$$\begin{aligned} -8u + 12 &\leq 20 \\ -8u + 12 - 12 &\leq 20 - 12 \\ -8u &\leq 8 \\ \frac{-8u}{-8} &\geq \frac{8}{-8} \\ u &\geq -1 \end{aligned}$$



Example

Ben can save \$180 each week, but he currently owes the bank \$630. He can afford to go on vacation once he has more than \$4500 saved in his bank account. When can Ben afford to go on vacation?

Choose and define the variable.

Let x be the number of weeks passed.

Write the problem as an inequality.

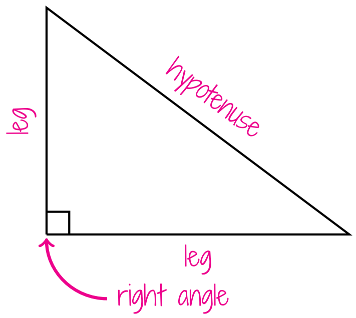
$180x - 630 > 4500$

Solve the inequality.

$$\begin{aligned} 180x - 630 &> 4500 \\ 180x - 630 + 630 &> 4500 + 630 \\ 180x &> 5130 \\ \frac{180x}{180} &> \frac{5130}{180} \\ x &> 28.5 \end{aligned}$$

Ben can afford his vacation after 29 or more weeks.

5.1 The Pythagorean Theorem



A right angle is an angle which measures 90°.

A right triangle is a triangle with a right angle.

The longest side of a right triangle is called the hypotenuse. The other two sides are called the legs. Notice that the legs are adjacent to the right angle (they touch it), while the hypotenuse is not.

THE PYTHAGOREAN THEOREM

Let a, b and c be the lengths of the sides of a triangle, where c is the longest side.

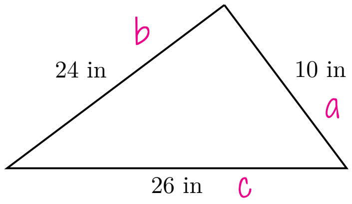
The triangle is a right triangle if and only if

$$a^2 + b^2 = c^2$$

This means that in a right triangle a and b are the lengths of the legs, and c is the length of the hypotenuse. It's always a good idea to label the sides a, b and c when working a right triangle problem.

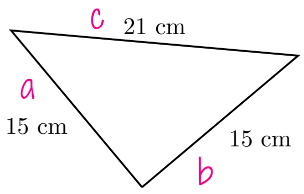
Example

Determine if the following triangles are right triangles.



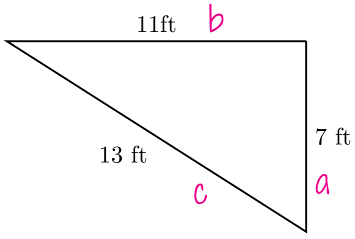
$$\begin{aligned} a^2 + b^2 &= 10^2 + 24^2 \\ &= 100 + 576 \\ &= 676 \\ c^2 &= 26^2 \\ &= 676 \\ a^2 + b^2 &= c^2 \end{aligned}$$

This is a right triangle.



$$\begin{aligned} a^2 + b^2 &= 15^2 + 15^2 \\ &= 225 + 225 \\ &= 450 \\ c^2 &= 21^2 \\ &= 441 \\ a^2 + b^2 &\neq c^2 \end{aligned}$$

This is NOT a right triangle.



$$\begin{aligned} a^2 + b^2 &= 7^2 + 11^2 \\ &= 49 + 121 \\ &= 170 \\ c^2 &= 13^2 \\ &= 169 \\ a^2 + b^2 &\neq c^2 \end{aligned}$$

This is NOT a right triangle.

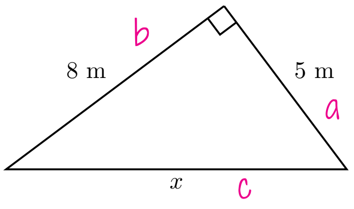
5.2 Lengths in Right Triangles

If we know that a triangle is a right triangle, and we know the lengths of two sides, we can find the length of the other side using the Pythagorean theorem.

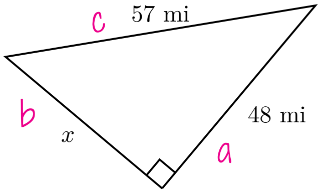
Don't forget that c is always assigned to the length of the hypotenuse, and that a and b are assigned to the legs. Always check that the hypotenuse works out to be the longest side.

Example

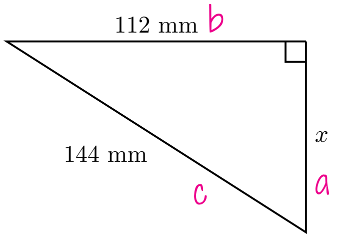
Find the length x in each triangle.



$$\begin{aligned} c^2 &= a^2 + b^2 \\ x^2 &= 5^2 + 8^2 \\ &= 25 + 64 \\ &= 89 \\ x &= \sqrt{89} \\ &= 9.43 \text{ m} \end{aligned}$$



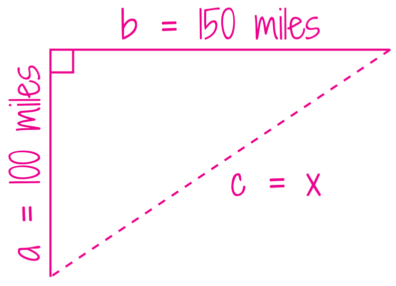
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 48^2 + x^2 &= 57^2 \\ x^2 &= 57^2 - 48^2 \\ &= 3249 - 2304 \\ &= 945 \\ x &= \sqrt{945} \\ &= 30.7 \text{ mi} \end{aligned}$$



$$\begin{aligned} a^2 + b^2 &= c^2 \\ x^2 + 112^2 &= 144^2 \\ x^2 &= 144^2 - 112^2 \\ &= 20736 - 12544 \\ &= 8192 \\ x &= \sqrt{8192} \\ &= 90.5 \text{ mm} \end{aligned}$$

Example

A ship sails 100 miles north from its dock, then turns east and sails 150 miles. How far is the ship from the dock?

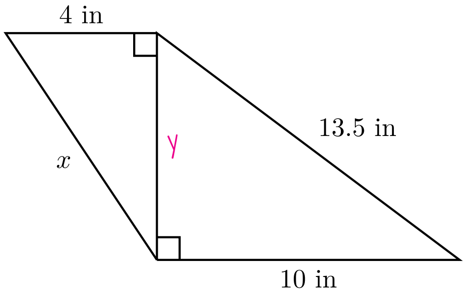


$$\begin{aligned} c^2 &= a^2 + b^2 \\ x^2 &= 100^2 + 150^2 \\ &= 10000 + 22500 \\ &= 32500 \\ x &= \sqrt{32500} \\ &= 180.28 \text{ mi} \end{aligned}$$

5.3 Multi-Step Right Triangle Problems

Example

Find the length x.

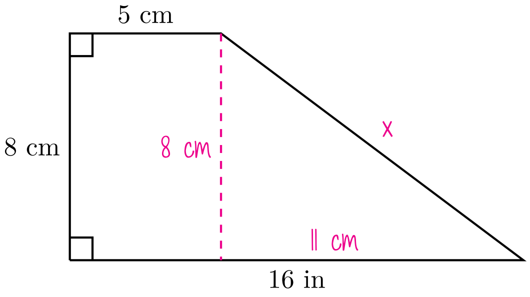


$$y^2 + 10^2 = 13.5^2$$
$$y^2 = 13.5^2 - 10^2$$
$$= 182.25 - 100$$
$$= 82.25$$
$$y = \sqrt{82.25}$$
$$= 9.07 \text{ in}$$

$$x^2 = 4^2 + y^2$$
$$= 16 + 82.25$$
$$= 98.25$$
$$x = \sqrt{98.25}$$
$$= 9.91 \text{ in}$$

Example

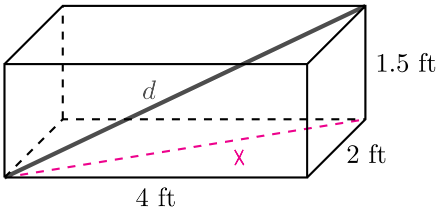
Find the perimeter of the trapezoid.



$$x^2 = 8^2 + 11^2$$
$$= 64 + 121$$
$$= 185$$
$$x = \sqrt{185}$$
$$= 13.60 \text{ cm}$$
$$P = 16 + 8 + 5 + 13.60$$
$$= 42.60 \text{ cm}$$

Example

Find the length of the diagonal d.



$$x^2 = 4^2 + 2^2$$
$$= 16 + 4$$
$$= 20$$
$$x = \sqrt{20}$$
$$= 4.47 \text{ ft}$$

$$d^2 = x^2 + 1.5^2$$
$$= 20 + 2.25$$
$$= 22.25$$
$$d = \sqrt{22.25}$$
$$= 4.71 \text{ ft}$$

5.4 Distances on the Coordinate Plane

Coordinate Plane Review

The coordinate plane represents the values of two variables with a point. Its horizontal position is value of x, and its vertical position is the value y.

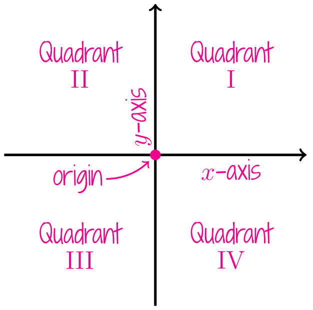
An ordered pair is written as (x, y). It too represents the values of the variables x and y, in that order, and the coordinates of a point on the plane.

The x-axis is the horizontal line where $y = 0$.

The y-axis is the vertical line where $x = 0$.

The origin is where the axes intersect, at the point $(0, 0)$.

The quadrants are the four regions separated by the axes.



Example

a) Write down the coordinates of A, B and C.

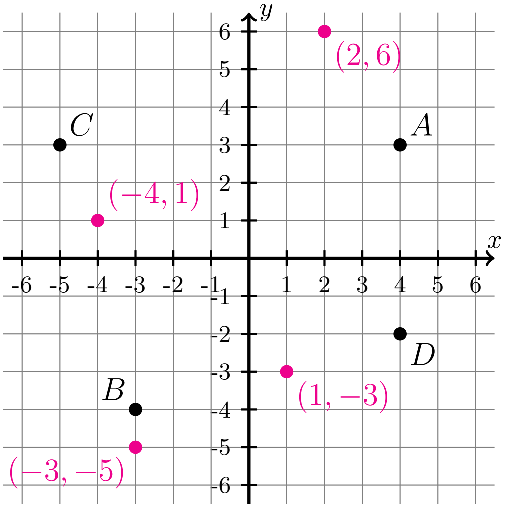
$A : (4, 3)$ $B : (-3, -4)$ $C : (-5, 3)$

b) What variable values does D represent?

$D : x = 4, y = 2$

c) Plot and label the points $(-4, 1)$, $(4, -3)$ and $(-3, -5)$.

d) Plot and label the point representing $x = 2$ and $y = 6$.



Calculating Distances

Example

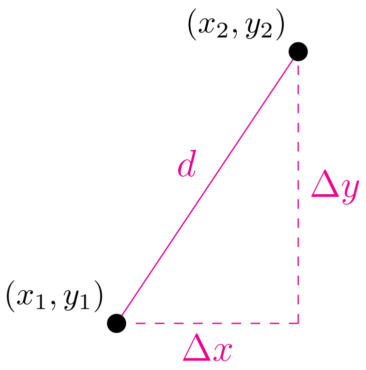
Consider the distances between A, B, C and D above. What are the two simplest distances to find?

Distance between A and C is 9. Distance between A and D is 5.

Why are these distances simpler to find than the others?

The points lay on the same horizontal or vertical line, which means the coordinate grid can be used to directly measure their distance.

The distance between two points is the same as the length of a line segment between them. We can form a right triangle with the distance as the hypotenuse and horizontal and vertical line segments as legs.



The lengths of these legs represent the changes in x and y between the two points. The Greek letter delta, Δ can be used to mean the change in a variable.

$$\Delta x = \text{change in } x = x_2 - x_1 \qquad \Delta y = \text{change in } y = y_2 - y_1$$

THE PYTHAGOREAN THEOREM for the distance between points d

$$d^2 = (\Delta x)^2 + (\Delta y)^2$$

Example

Find the distance between $(2, 3)$ and $(5, -4)$.

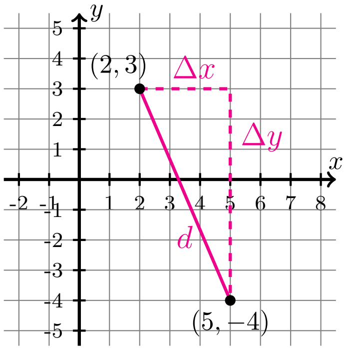
	x	y	
$+3$	2	3	-7
	5	-4	

$\Delta x = 3$

$\Delta y = -7$

$$\begin{aligned} d^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (3)^2 + (-7)^2 \\ &= 9 + 49 \\ &= 58 \end{aligned}$$

$$d = \sqrt{58} = 7.62$$



Find the distance between $(1, -4)$ and $(7, -2)$.

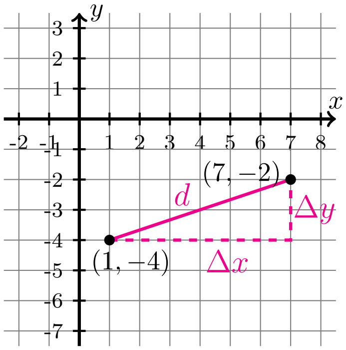
	x	y	
$+6$	1	-4	$+2$
	7	-2	

$\Delta x = 6$

$\Delta y = 2$

$$\begin{aligned} d^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (6)^2 + (2)^2 \\ &= 36 + 4 \\ &= 40 \end{aligned}$$

$$d = \sqrt{40} = 6.32$$



6.1 Function Rules and Tables

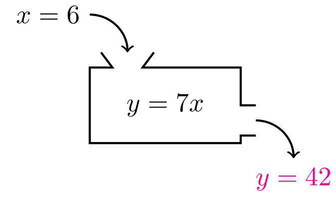
A relation is a collection of ordered pairs which represents a relationship between two variables.

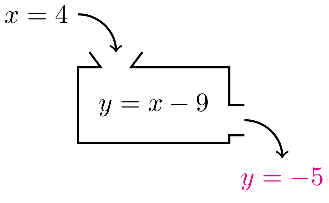
A function is a relation where the value of the independent variable, usually x, determines the value of the dependent variable, usually y. Each input (x value) in a function produces exactly one output (y value).

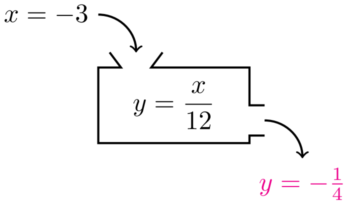
Two ways to represent functions are algebraic rules and tables.

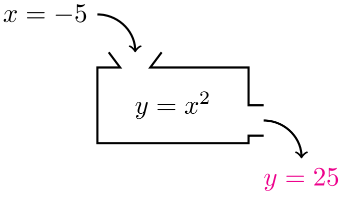
Example

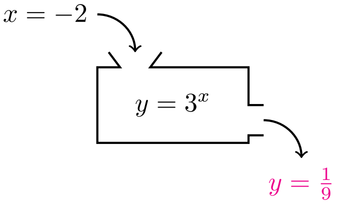
Write in the output for the following function machines.

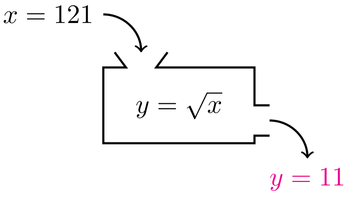












Example

For the function with the rule $y = 2x + 5$, determine the output for each input.

$x = 3$

$$\begin{aligned} y &= 2(3) + 5 \\ &= 6 + 5 \\ &= 11 \end{aligned}$$

$x = -6$

$$\begin{aligned} y &= 2(-6) + 5 \\ &= -12 + 5 \\ &= -7 \end{aligned}$$

$x = 7.5$

$$\begin{aligned} y &= 2(7.5) + 5 \\ &= 15 + 5 \\ &= 20 \end{aligned}$$

For the function with the rule $y = x^2 - 9$, determine the output for each input.

$x = 1$

$$\begin{aligned} y &= (1)^2 - 9 \\ &= 1 - 9 \\ &= -8 \end{aligned}$$

$x = -3$

$$\begin{aligned} y &= (-3)^2 - 9 \\ &= 9 - 9 \\ &= 0 \end{aligned}$$

$x = 4.5$

$$\begin{aligned} y &= (4.5)^2 - 9 \\ &= 20.25 - 9 \\ &= 11.25 \end{aligned}$$

Example

Complete the table for the function $y = 4x - 11$.

input	x	-3	-2	-1	0	1	2	3	4	5
output	y	-23	-19	-15	-11	-7	-3	1	5	9

Complete the table for the function $y = -3x + 5$.

input	x	-6	-4	-3	0	1	2	5	7	10
output	y	23	17	14	5	2	-1	-10	-16	-25

Complete the table for the function $y = x^3$.

input	x	-4	-3	-2	-1	0	1	2	3	4
output	y	-64	-27	-8	-1	0	1	8	27	64

6.2 Finding Linear Rules from Tables

A linear function is a function whose output results from multiplying the input by a constant and adding another constant. All linear functions can be written in the same general form.

LINEAR FUNCTION GENERAL FORM

$$y = mx + b$$

where m and b are constant.

Example

Find the constants m and b for these linear functions.

$y = -3x + 7$

$m = -3, b = 7$

$y = \frac{x}{4} - 9$

$m = \frac{1}{4}, b = -9$

$y = 9x$

$m = 9, b = 0$

$y = 13 - 7x$

$y = -7x + 13$

$m = -7, b = 13$

$y = -4(x - 5)$

$y = -4x + 20$

$m = -4, b = 20$

$y = \frac{3x+4}{6}$

$y = \frac{1}{2}x + \frac{2}{3}$

$m = \frac{1}{2}, b = \frac{2}{3}$

The rate of change between two points of a function is the ratio of the change in the output and the change in the input.

$$\text{rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

Example

Complete the table for each function. Then find the rate of change between each pair of points.

	input	output	
	x	y	
	0	-6	
+1			+4
	1	-2	
+1			+4
	2	2	
+1			+4
	3	6	
+1			+4
	4	10	

$y = 4x - 2$

$\Delta x = 1$

$\Delta y = 4$

$\frac{\Delta y}{\Delta x} = \frac{4}{1}$

$= 4$

	input	output	
	x	y	
	-3	7	
+3			-2
	0	5	
+3			-2
	3	3	
+3			-2
	6	1	
+3			-2
	9	-1	

$y = -\frac{2}{3}x + 5$

$\Delta x = 3$

$\Delta y = -2$

$\frac{\Delta y}{\Delta x} = \frac{-2}{3}$

$= -\frac{2}{3}$

What do you notice? What do you wonder?

THE RATE OF CHANGE OF A LINEAR FUNCTION

Linear functions are functions with a constant rate of change**,**

which is the coefficient of x **in the general form.**

$$m = \frac{\Delta y}{\Delta x} \qquad \text{or} \qquad \Delta y = m \cdot \Delta x$$

Example

Find a rule for the linear function described in each table.

	input	output	
	x	y	
	0	7	
+1			+5
	1	12	
+1			+5
	2	17	
+1			+5
	3	22	

$\Delta x = 1$ and $\Delta y = 5$

$$m = \frac{\Delta y}{\Delta x} = \frac{5}{1} = 5$$

When $x = 0$, $y = 7$.

$y = mx + b$

$7 = 5(0) + b$

$7 = 0 + b$

$b = 7$

So, $y = 5x + 7$

	input	output	
	x	y	
	−4	5	
+3			−6
	−1	−1	
+2			−4
	1	−5	
+4			−8
	5	−13	

$\Delta x = 6$ and $\Delta y = -12$

$$m = \frac{\Delta y}{\Delta x} = \frac{-12}{6} = -2$$

When $x = 5$, $y = -13$.

$y = mx + b$

$-13 = -2(5) + b$

$-13 = -10 + b$

$b = -3$

So, $y = -2x - 3$

	input	output	
	x	y	
	0	8	
+8			+6
	8	14	
+4			+3
	12	17	
+12			+9
	24	26	

$\Delta x = 8$ and $\Delta y = 6$

$$m = \frac{\Delta y}{\Delta x} = \frac{6}{8} = \frac{3}{4}$$

When $x = 0$, $y = 8$.

$y = mx + b$

$8 = \frac{3}{4}(0) + b$

$8 = 0 + b$

$b = 8$

So, $y = \frac{3}{4}x + 8$

To find a rule from linear table:

Step 1. Use the table to calculate the rate of change. This is m.

Step 2. Using m and the values for x and y from one point, solve for b.

Step 3. Use m and b to write down the rule.

Step 4. Check that the rule is true for the values in the table.

6.3 Plotting Function Graphs

An ordered pair, written as (x, y) has two equivalent meanings:

- The values of the two variables, x and y, in that order.
- The coordinates of a point on the coordinate plane.

A function describes a relationship between values of x and values of y. This means we can represent a function by plotting a graph on the coordinate plane.

Example

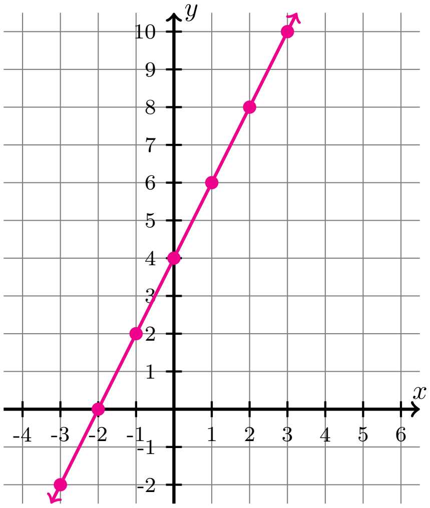
Complete the table for the function $y = 2x + 4$.

x	-3	-2	-1	0	1	2	3
y	-2	0	2	4	6	8	10

Write the entries from the table as a list of ordered pairs.

$(-3,-2), (-2,0), (-1,2), (0,4), (1,6),$
$(2,8), (3,10)$

Plot a graph of the function on the coordinate plane.

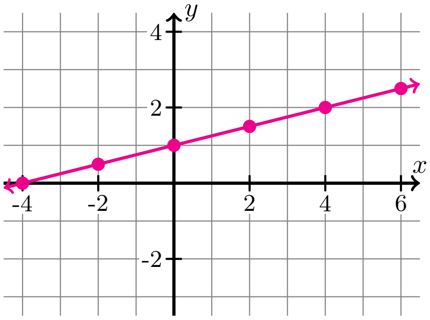


Example

Complete the table for the function $y = \frac{x}{4} + 1$.

x	-4	-2	0	2	4	6
y	0	0.5	1	1.5	2	2.5

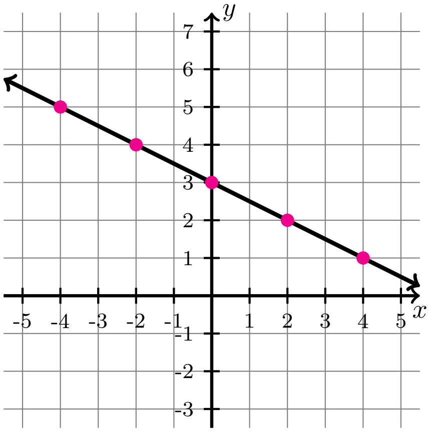
Plot a graph of the function on the coordinate plane.



A function of the form $y = mx + b$ is called a linear function **because its graph is a** straight line.

Example

Complete the tables and find the rules for the functions shown in the graphs.



	x	y	
	-4	5	
$+2$			-1
	-2	4	
$+2$			-1
	0	3	
$+2$			-1
	2	2	
$+2$			-1
	4	1	

$\Delta x = 2, \Delta y = -1$

$m = \dfrac{\Delta y}{\Delta x} = \dfrac{-1}{2} = -\frac{1}{2}$

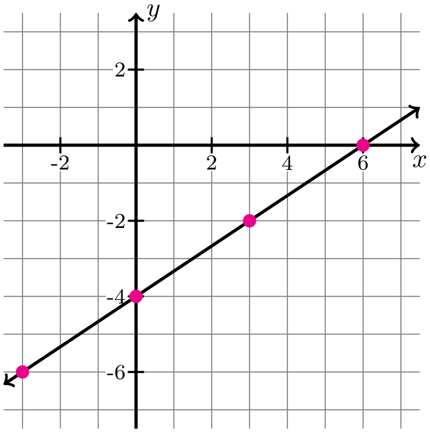
When $x = 0,\ y = 3.$

$y = mx + b$

$3 = -\frac{1}{2}(0) + b$

$b = 3$

$y = -\frac{1}{2}x + 3$



	x	y	
	-3	-6	
$+3$			$+2$
	0	-4	
$+3$			$+2$
	3	-2	
$+3$			$+2$
	6	0	

$\Delta x = 3, \Delta y = 2$

$m = \dfrac{\Delta y}{\Delta x} = \dfrac{2}{3}$

When $x = 0,\ y = -4.$

$y = mx + b$

$-4 = \frac{2}{3}(0) + b$

$-4 = 0 + b$

$b = -4$

$y = \frac{2}{3}x - 4$

6.4 Identifying Linear and Nonlinear Functions

A nonlinear function is a function which is not a linear function.

LINEAR FUNCTIONS vs. NONLINEAR FUNCTIONS

	linear functions	nonlinear functions
rule	can be written as $y = mx + b$	can't be written as $y = mx + b$
table	constant rate of change	changing rate of change
plot	straight line	not a straight line

Example

Does the rule $y = -\frac{3}{2}(x + 4) + 11$ represent a linear function?

$$\begin{aligned} y &= -\tfrac{3}{2}(x+4) + 11 \\ &= -\tfrac{3}{2}x - 6 + 11 \\ &= -\tfrac{3}{2}x + 5 \end{aligned}$$

The rule can be written in the form $y = mx + b$, which means that it represents a linear function.

Complete the table for the function above. Does this show a linear function?

	x	y	
	-4	11	
$+4$			-6
	0	5	
$+2$			-3
	2	2	
$+6$			-9
	8	-7	

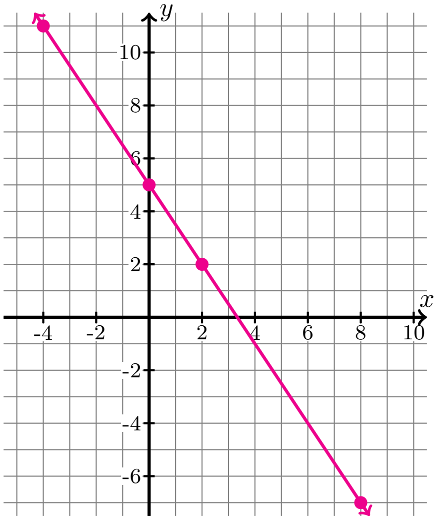
$$m = \frac{\Delta y}{\Delta x}$$

$$m_1 = \frac{-6}{4} = -\frac{3}{2}$$

$$m_2 = \frac{-3}{2} = -\frac{3}{2}$$

$$m_3 = \frac{-9}{6} = -\frac{3}{2}$$

The table has the same rate of change between each pair of points, which means that it represents a linear function.



Plot the function above on the coordinate plane.
Does this show a linear function?

The plot forms a straight line, which means that it represents a linear function.

Example

Does the rule $y = x^2$ represent a linear function?

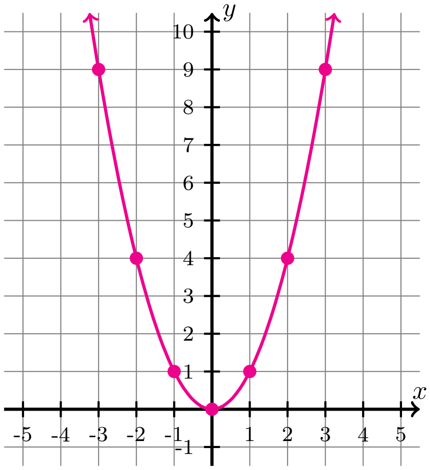
There is no equivalent to $y = x^2$ that can be written using $y = mx + b$ because there is a power of 2 on the x. This is a nonlinear function.

Complete the table for the function above. Does this show a linear function?

	x	y	
	-3	9	
$+1$			-5
	-2	4	
$+1$			-3
	-1	1	
$+1$			-1
	0	0	
$+1$			$+1$
	1	1	
$+1$			$+3$
	2	4	
$+1$			$+5$
	3	9	

$$m = \frac{\Delta y}{\Delta x}$$
$$m_1 = \frac{-5}{1} = -5$$
$$m_2 = \frac{-3}{1} = -3$$
$$m_3 = \frac{-1}{1} = -1$$
$$m_4 = \frac{1}{1} = 1$$
$$m_5 = \frac{3}{1} = 3$$
$$m_6 = \frac{5}{1} = 5$$

The rate of change is not the same between each pair of points, which means that the table represents a nonlinear function.



Plot the function above on the coordinate plane.
Does this show a linear function?

The points on the plot do not form a straight line, which means that it represents a nonlinear function.

7.1 Intercepts

In a graph, an intercept is a point where a function crosses an axis.

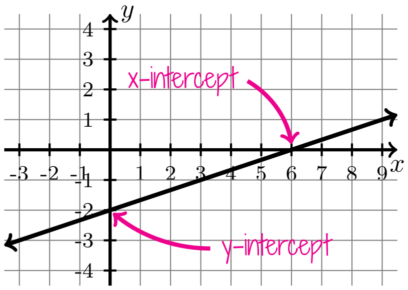
An intercept on the x-axis is an x-intercept, and on the y-axis is a y-intercept.

Example

State the intercepts of the graph.

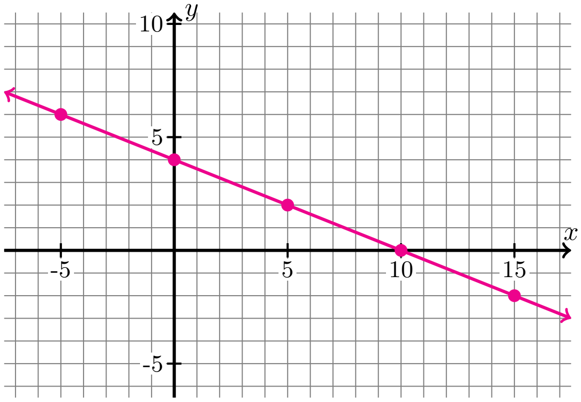
x-intercept is $(6, 0)$

y-intercept is $(0, -2)$



Complete the table and plot for $y = -\frac{2x}{5} + 4$.

x	-5	0	5	10	15
y	6	4	2	0	-2



State the intercepts of the graph.

x-intercept is $(10, 0)$

y-intercept is $(0, 4)$

What do you notice? What do you wonder?

x-intercepts occur when $y = 0$. **y-intercepts occur when** $x = 0$.

Example

Find the intercepts of the graph of $y = \frac{2}{3}x + 8$.

x-intercept: $y = 0$

$$\frac{2}{3}x + 8 = 0$$
$$\frac{2}{3}x = -8$$
$$x = -8 \cdot \frac{3}{2}$$
$$x = -12$$

y-intercept: $x = 0$

$$y = \frac{2}{3}(0) + 8$$
$$= 0 + 8$$
$$= 8$$

x-intercept at $(-12, 0)$

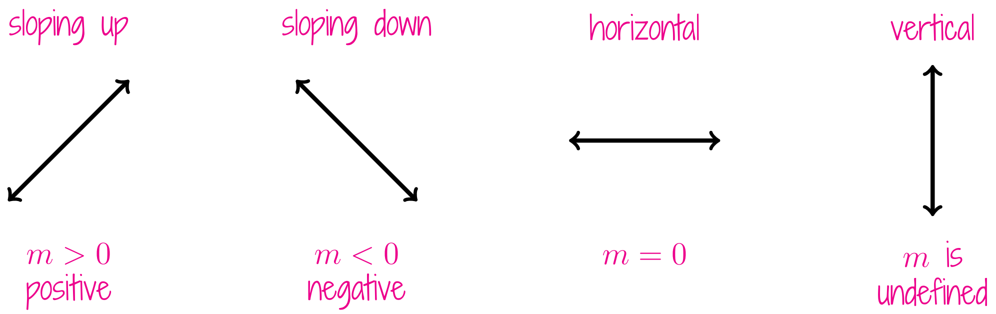
y-intercept at $(0, 8)$

7.2 Slope

The slope of a line is a measure of its direction and steepness. Slope is calculated as the ratio of the vertical distance to the horizontal distance between two points on the line.

The SLOPE of the graph of a LINEAR FUNCTION is identical to the function's rate of change**.**

$$m = \frac{\Delta y}{\Delta x} \quad \text{or} \quad \Delta y = m \cdot \Delta x$$



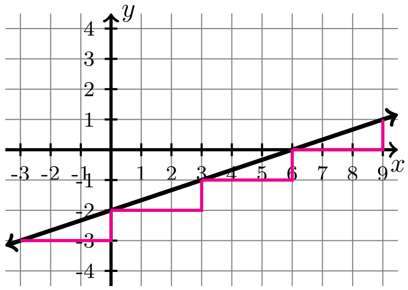
Example

Calculate the slope of the line.

The line goes 1 unit up for every 3 units right.

$\Delta y = 1$ and $\Delta x = 3$.

$$m = \frac{\Delta y}{\Delta x} = \frac{2}{6} = \frac{1}{3}$$



Example

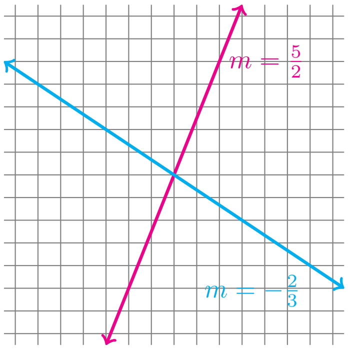
Describe the direction of a line with slope $m = \frac{5}{2}$.

The line goes 5 units up for every 2 units right.

Describe the direction of a line with slope $m = -\frac{2}{3}$.

The line goes 2 units down for every 3 units right.

Draw an example of each slope on the grid provided.



Example

Plot the line which passes through the point $(5,6)$ with slope 2.

What are the intercepts of this line?

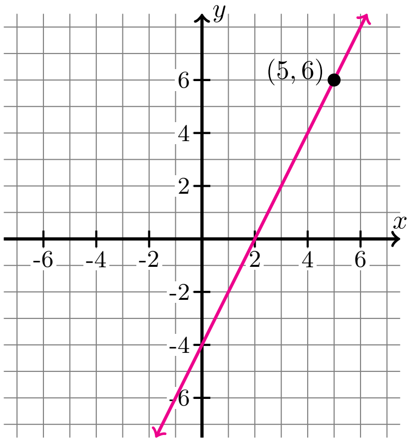
x-intercept is $(2,0)$, y-intercept is $(-4,0)$

The point $(9,k)$ is also on the line. What is k?

From $(5,6)$ to $(9,k)$, $\Delta x = 4$

$\Delta y = m \cdot \Delta x = 2 \cdot 4 = 8$

$k = 6 + 8 = 14$



Example

What is the slope of the graph of $y = -\frac{1}{3}x + 2$?

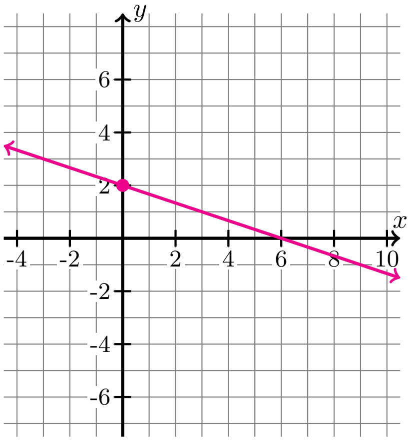
$m = -\frac{1}{3}$

What is the y-intercept? Plot it on the coordinate plane.

$y = -\frac{1}{3}(0) + 2 = 2$

y-intercept is $(0,2)$

Plot a graph of the function by drawing a line from the y-intercept with the correct slope.



7.3 Slope-Intercept Form

We have already learned that:

- The slope of a graph is the same as the rate of change of the function.
- The y-intercept is the point where the function's input is $x = 0$.
- The x-intercept is the point where the function's output is $y = 0$.

SLOPE-INTERCEPT FORM

is the general form of a linear function $y = mx + b$,

because m is the slope of the graph

and $(0, b)$ is the y-intercept of the graph.

A sketch is a type of graph which only shows the most important information of a function, such as intercepts. A sketch must be neat, using a ruler for straight lines.

Example

Find the intercepts and the slope, then sketch the graph, of the function $y = -4x + 8$.

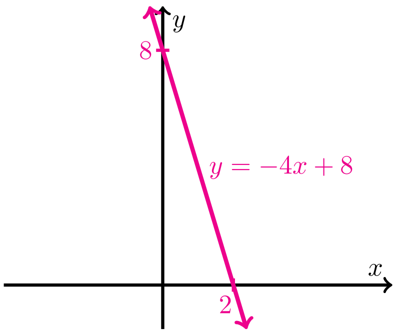
x-intercept: $(2, 0)$

y-intercept: $(0, 8)$
because $b = 8$

Slope: $m = -4$

Solve $mx + b = 0$:

$$-4x + 8 = 0$$
$$-4x + 8 - 8 = 0 - 8$$
$$-4x = -8$$
$$\frac{-4x}{-4} = \frac{-8}{-4}$$
$$x = 2$$



Find the intercepts and the slope, then sketch the graph, of the function $y = 2x - 6$.

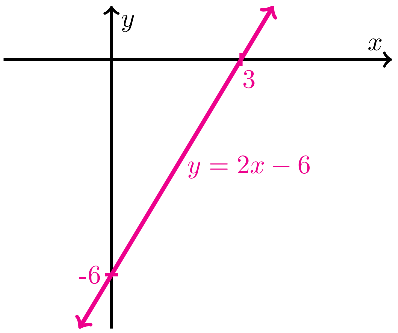
x-intercept: $(3, 0)$

y-intercept: $(0, -6)$
because $b = -6$

Slope: $m = 2$

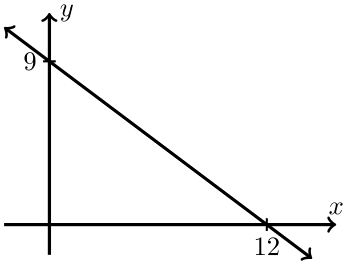
Solve $mx + b = 0$:

$$2x - 6 = 0$$
$$2x - 6 + 6 = 0 + 6$$
$$2x = 6$$
$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$



Example

Find the rule for the function in the sketch.

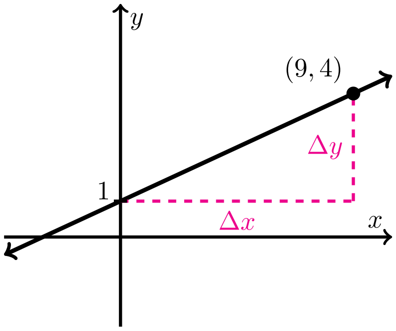


$$+12\ \begin{array}{c|c} x & y \\ \hline 0 & 9 \\ 12 & 0 \end{array}\ -9$$

$$m = \frac{-9}{12}$$
$$= -\frac{3}{4}$$
$$b = 9$$

$$y = -\frac{3}{4}x + 9$$

Find the rule for the function in the sketch, and find the location of the unlabeled x-intercept.



$$+9\ \begin{array}{c|c} x & y \\ \hline 0 & 1 \\ 9 & 4 \end{array}\ +3$$

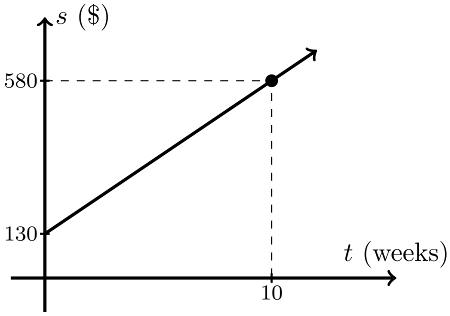
$$m = \frac{3}{9}$$
$$= \frac{1}{3}$$
$$b = 1$$
$$y = \frac{1}{3}x + 1$$

$$\frac{1}{3}x + 1 = 0$$
$$\frac{1}{3}x = -1$$
$$3 \cdot \frac{1}{3}x = 3(-1)$$
$$x = -3$$

x-intercept is $(-3, 0)$

Example

Melanie has a savings account she is using to save up to buy a computer for \$850. Her savings balance since the start of the year is shown in the graph.



What does the independent variable represent?

t is the time passed since the start of the year, in weeks.

What does the dependent variable represent?

s is the balance of the savings account, in dollars.

What does the marked point represent?

\$580 was saved after 10 weeks.

What does the s-intercept represent?

The intercept is $(0, 130)$. This shows that the balance was \$130 at the start of the year.

What is the slope of the graph? What does this represent?

$$+10\ \begin{array}{c|c} x & y \\ \hline 0 & 130 \\ 10 & 580 \end{array}\ +450$$

$$m = \frac{450}{10} = 45$$

is the amount of money saved each week.

Find the rule for the function representing Melanie's savings.

$s = 45t + 130$

When will Melanie's savings be enough for the computer?

$$45t + 130 = 850$$
$$45t = 720$$
$$t = \frac{720}{45} = 16 \text{ weeks}$$

7.4 Finding Linear Rules from Points

To write down a rule in slope-intercept form, we need to know the slope and the y-intercept. Sometimes, we need to use other points to find these.

Example

Find the y-intercept and the rule for each of the described lines.

Slope $m = -3$, passing through $(4, -1)$.

$$y = -3x + b$$

When $x = 4, y = -1$

$$-1 = -3(4) + b$$
$$-1 = -12 + b$$
$$-1 + 12 = -12 + 12 + b$$
$$b = 11$$

y-intercept is $(0, 11)$

rule is $y = -3x + 11.$

Slope $m = \frac{2}{5}$, x-intercept at $x = -10$.

$$y = \tfrac{2}{5}x + b$$

When $x = -10, y = 0$

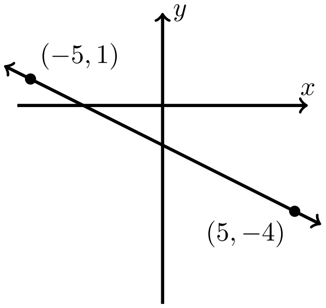
$$0 = \tfrac{2}{5}(-10) + b$$
$$0 = -4 + b$$
$$0 + 4 = -4 + 4 + b$$
$$b = 4$$

y-intercept is $(0, 4)$

rule is $y = \frac{2}{5}x + 4.$

Example

Find a rule for the line shown.



$$+10 \; \begin{array}{c|c} x & y \\ \hline -5 & 1 \\ 5 & -4 \end{array} \; -5$$

$$m = \frac{\Delta y}{\Delta x} = \frac{-5}{10}$$
$$= -\tfrac{1}{2}$$

$$y = -\tfrac{1}{2}x + b$$

When $x = 5,\ y = -4.$

$$-4 = -\tfrac{1}{2}(5) + b$$
$$-4 = -\tfrac{5}{2} + b$$
$$-4 + \tfrac{5}{2} = -\tfrac{5}{2} + \tfrac{5}{2} + b$$
$$b = \tfrac{3}{2}$$

$$y = -\tfrac{1}{2}x + \tfrac{3}{2}$$

Find a rule for the line which passes through the points $(-1, 5)$ and $(7, 9)$.

$$+8 \; \begin{array}{c|c} x & y \\ \hline -1 & 5 \\ 7 & 9 \end{array} \; +4$$

$$m = \frac{\Delta y}{\Delta x} = \frac{4}{8}$$
$$= \tfrac{1}{2}$$
$$y = \tfrac{1}{2}x + b$$

When $x = 7,\ y = 9.$

$$9 = \tfrac{1}{2}(7) + b$$
$$9 = \tfrac{7}{2} + b$$
$$9 - \tfrac{7}{2} = \tfrac{7}{2} - \tfrac{7}{2} + b$$
$$b = \tfrac{11}{2}$$

$$y = \tfrac{1}{2}x + \tfrac{11}{2}$$

Point-Slope Form

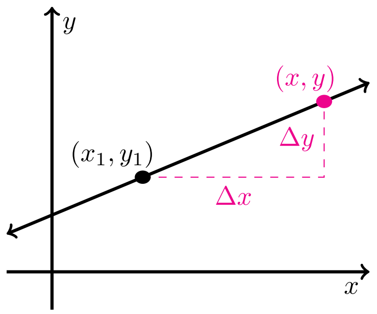
Suppose a particular point (x_1, y_1) is on a line. We can use (x, y) to represent every point on the line. The changes between the points are

$$\Delta x = x - x_1 \qquad \Delta y = y - y_1$$

If the line has slope m, then $\Delta y = m \cdot \Delta x$.

$$y - y_1 = m(x - x_1)$$

$$y = m(x - x_1) + y_1$$



The POINT-SLOPE FORM of a line with slope m passing through (x_1, y_1) is

$$y = m(x - x_1) + y_1$$

Example

a) Write rules for these lines in point-slope form.

Slope $m = -2$, passing through $(-5, 7)$.

$y = m(x - x_1) + y_1$

$y = -2(x - (-5)) + 7$

$y = -2(x + 5) + 7$

Slope $m = \frac{3}{4}$, passing through $(8, -2)$.

$y = m(x - x_1) + y_1$

$y = \frac{3}{4}(x - 8) + (-2)$

$y = \frac{3}{4}(x - 8) - 2$

b) Write each rule in slope-intercept form.

$y = -2x - 10 + 7$

$y = -2x - 3$

$y = \frac{3}{4}x - 6 - 2$

$y = \frac{3}{4}x - 8$

Example

Find the rule for this line in both point-slope and slope-intercept forms.

$$\begin{array}{cc|cc} & x & y & \\ \hline +1 & 1 & -1 & +4 \\ & 2 & 3 & \end{array}$$

$m = \frac{4}{1} = 4$

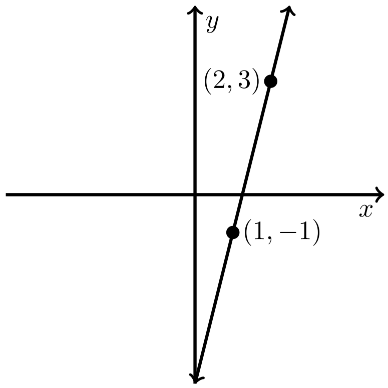
$x_1 = 2$

$y_1 = 3$

$y = 4(x - 2) + 3$ (point-slope form)

$= 4x - 8 + 3$

$y = 4x - 5$ (slope-intercept form)



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7.5 Standard Form

The STANDARD FORM of the equation of a line is

$$Ax + By = C$$

- **Constants** A, B **and** C **are** integers**, if possible.**
- A **is** non-negative**.**
- **The equation is** simplified**, so** A, B **and** C **have no common** factors**.**

Example

Do the points $(6, 5)$ and $(-2, 4)$ lie on the line $5x - 2y = 20$?

$x = 6, y = 5$

$$\begin{aligned} 5x - 2y &= 5(6) - 2(5) \\ &= 30 - 10 \\ &= 20 \end{aligned}$$

The equation is true, so $(6, 5)$ is on the line.

$x = -2, y = 4$

$$\begin{aligned} 5x - 2y &= 5(-2) - 2(4) \\ &= -10 - 8 \\ &= -18 \end{aligned}$$

The equation is false, so $(-2, 4)$ is not on the line.

Remember that x-intercepts occur when $y = 0$, and y-intercepts occur when $x = 0$.

Example

Sketch a graph and find the slope of $x + 2y = 6$.

x-intercept: $(6, 0)$

$$\begin{aligned} y &= 0 \\ x + 2(0) &= 6 \\ x &= 6 \end{aligned}$$

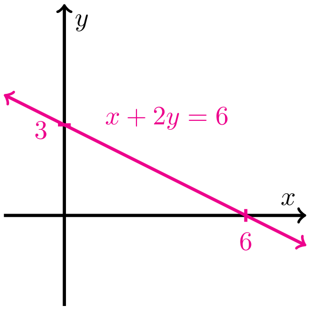
y-intercept: $(0, 3)$

$$\begin{aligned} x &= 0 \\ (0) + 2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

slope:

$$\begin{array}{r|c|c|l} & x & y & \\ \hline +6 & 0 & 3 & -3 \\ & 6 & 0 & \end{array}$$

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{3}{-6} \\ &= -\frac{1}{2} \end{aligned}$$



To find the slope for an equation in standard form, we can use the intercepts to calculate it, or we can convert the equation to slope-intercept form.

Example

Check the results of the previous example by writing $x + 2y = 6$ in slope-intercept form.

$$\begin{aligned} x + 2y &= 6 \\ x - x + 2y &= 6 - x \\ 2y &= -x + 6 \\ \frac{2y}{2} &= \frac{-x+6}{2} \\ y &= -\tfrac{1}{2}x + 3 \end{aligned}$$

The slope is $m = -\frac{1}{2}$.

The y-intercept is $(0, 3)$.

These results match the answers from the previous example.

Find the slope of $3x - 4y = 8$ by writing the rule in slope intercept form.

$$\begin{aligned} 3x - 4y &= 8 \\ -3x + 3x - 4y &= -3x + 8 \\ -4y &= -3x + 8 \\ \frac{-4y}{-4} &= \frac{-3x+8}{-4} \\ y &= \tfrac{3}{4}x - 2 \end{aligned}$$

The slope is $m = \dfrac{3}{4}$.

Example

Convert these linear functions to standard form.

$y = \frac{2}{3}x - \frac{5}{6}$

$$\begin{aligned} 6 \cdot y &= 6 \cdot \tfrac{2}{3}x - 6 \cdot \tfrac{5}{6} \\ 6y &= 4x - 5 \\ -4x + 6y &= -4x + 4x - 5 \\ -4x + 6y &= -5 \\ 4x - 6y &= 5 \end{aligned}$$

$y = \frac{3}{4}x + \frac{5}{2}$

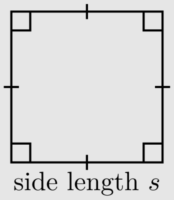
$$\begin{aligned} 4 \cdot y &= 4 \cdot \tfrac{3}{4}x + 4 \cdot \tfrac{5}{2} \\ 4y &= 3x + 10 \\ -3x + 4y &= -3x + 3x + 10 \\ -3x + 4y &= 10 \\ 3x - 4y &= -10 \end{aligned}$$

8.1 Perimeter and Area Review

The perimeter of a closed figure is the total length of its boundary. The area of a closed figure is a measure of the two-dimensional space contained in its interior.

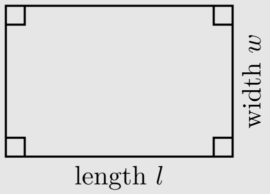
AREA of select two dimensional figures

square



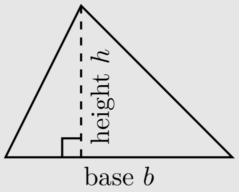
$A = s^2$

rectangle



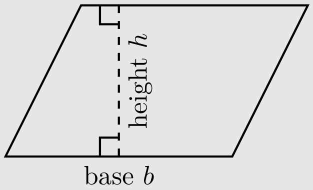
$A = lw$

triangle



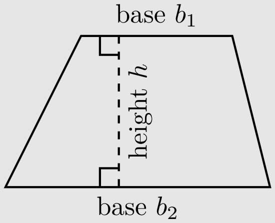
$A = \frac{1}{2}bh$

parallelogram



$A = bh$

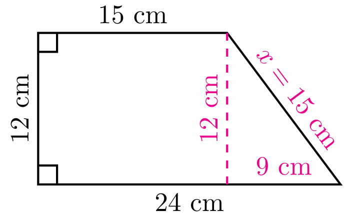
trapezoid



$$A = \frac{b_1 + b_2}{2} \cdot h$$

Example

Find the area and the perimeter of the following figure.



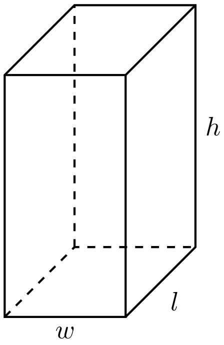
$b_1 = 15, b_2 = 24, h = 12$

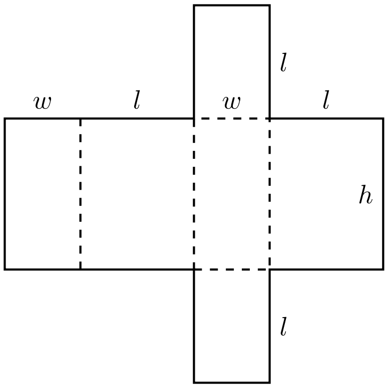
$$\begin{aligned} A &= \frac{b_1 + b_2}{2} \cdot h \\ &= \frac{15 + 24}{2} \cdot 12 \\ &= 19.5 \cdot 12 \\ &= 324 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} x^2 &= 9^2 + 12^2 \\ &= 225 \\ x &= \sqrt{225} \\ &= 15 \text{ cm} \\ P &= 24 + 12 + 15 + 15 \\ &= 66 \text{ cm} \end{aligned}$$

8.2 Prism Surface Area

A prism is a three-dimensional figure whose faces are two identical bases, connected on each edge by rectangles which are called the lateral faces. If the bases are also rectangles, the shape is a rectangular prism. If each rectangle is a square, the shape is a cube.



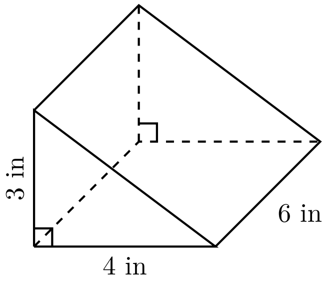


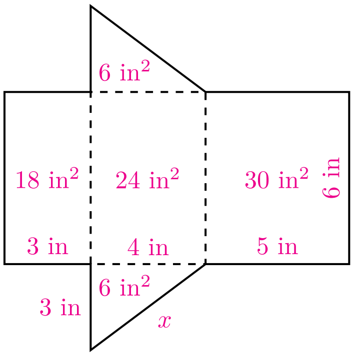
The surface area of a 3D shape is the sum of the areas of its faces.

A net is a 2D representation of a 3D shape that forms the shape when folded along its edges. The surface area of a 3D shape is the same as the area of its net.

Example

Use the net to find the surface area of the rectangular prism.





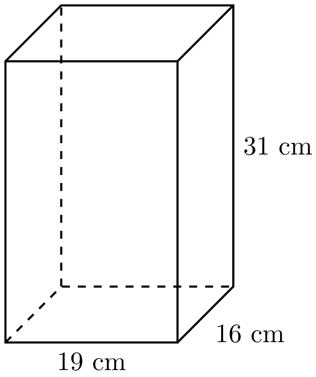
$$x^2 = 3^2 + 4^2$$
$$= 25$$
$$x = 5\text{cm}$$
$$S = (3+4+5)\cdot 6 + 2\left(\tfrac{1}{2}\cdot 3\cdot 4\right)$$
$$= 12\cdot 6 + 2\cdot 6$$
$$= 72 + 12$$
$$= 84 \text{ in}^2$$

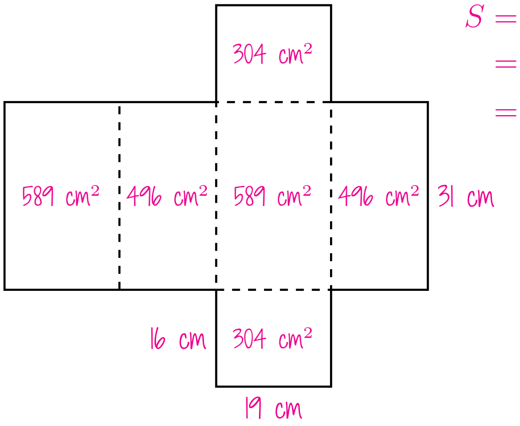
The SURFACE AREA OF A PRISM whose base has area B and perimeter P, and height is h

$$S = \text{area of bases} + \text{lateral area}$$
$$= 2B + Ph$$

Example

Use the net to find the surface area of the rectangular prism.





$$
\begin{aligned}
S &= 2(589 + 496 + 304) \\
&= 2(1389) \\
&= 2778 \text{ cm}^2
\end{aligned}
$$

The SURFACE AREA OF A RECTANGULAR with length l, width w, and height h

$$
\begin{aligned}
S &= 2lw + 2lh + 2wh \\
&= 2(lw + lh + wh)
\end{aligned}
$$

Example

Find the surface area of a cube with a side length of 11 inches.

A cube has 6 identical faces. The area of each face is $11^2 = 121$ in^2.

$$
\begin{aligned}
S &= 6 \cdot 121 \\
&= 726 \text{ in}^2
\end{aligned}
$$

8.3 Prism Volume

The volume of a 3D shape is a measure of the amount of three dimensional space it occupies. The volume of a prism is the product of the area of its base, and its height.

In a rectangular prism, the area of the base is the product of its length and width. This means that its volume can be found by multiplying its width, length, and height.

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The VOLUME OF A PRISM with base area B and height h

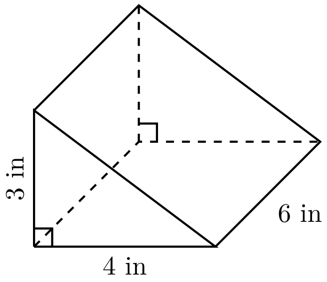
$$V = Bh$$

The VOLUME OF A RECTANGULAR PRISM with length l, width w, and height h

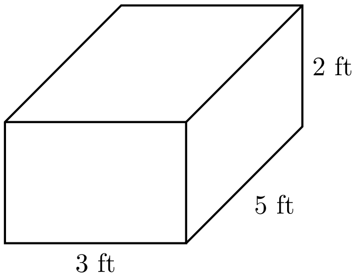
$$V = lwh$$

Example

Find the volumes of the prisms shown.



$$B = \frac{1}{2} \cdot 3 \cdot 4$$
$$= 6 \text{ in}^2$$
$$V = Bh$$
$$= 6 \cdot 6$$
$$= 36 \text{ in}^3$$



$$l = 5, \quad w = 3, \quad h = 2$$
$$V = lwh$$
$$= 5 \cdot 3 \cdot 2$$
$$= 30 \text{ ft}^3$$

Example

A prism with a square base has a height of 5 mm and a volume of 80 mm^3. What is the width of the prism?

Because the base is square, the width and the length are the same.

$$l = w, \quad h = 5, \quad V = 80$$
$$V = lwh$$
$$80 = w \cdot w \cdot 5$$
$$5w^2 = 80$$
$$\frac{5w^2}{5} = \frac{80}{5}$$
$$w^2 = 16$$
$$w = \sqrt{16}$$
$$= 4 \text{ mm}$$

Example

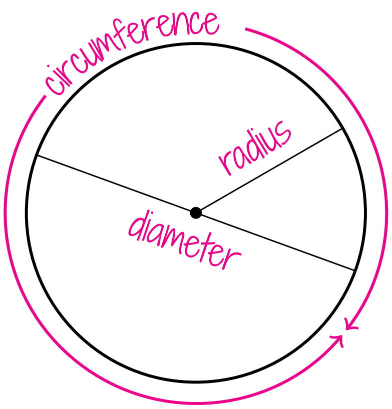
A rectangular prism has a width of 4 in and a height of 7 in. The volume of the prism, in inches, is irrational. What can you say about the length of the prism?

The volume of the prism is 28 times the length of the prism.

The product of two rational numbers is rational, but this product is not rational.

Since 28 is rational, the length must be an irrational number.

8.4 Circles Review



A circle is a 2D shape such that all its points are the same distance from its center.

A radius of a circle is a line segment between the center and a point on the circle. Its length is also called the radius.

A diameter of a circle is a line segment between opposite points on the circle, which passes through the center. Its length, which is double the radius, is also called the diameter.

The circumference is the curved length around the circle.

π, the Greek letter pi, is the ratio of the circumference to the diameter in every circle. Its value is an irrational number which can be approximated using $\pi \approx 3.14$.

The CIRCUMFERENCE C of a circle with radius r and diameter $d = 2r$

$$C = 2\pi r = \pi d$$

The AREA A of the interior of a circle with radius r

$$A = \pi r^2$$

Example

Find the circumference and area of a circle whose diameter is 6 in. Give answers exactly, and to two decimal places.

$$d = 6$$
$$r = 3$$

$$\begin{aligned} C &= \pi d \\ &= 6\pi \text{ in} \\ &\approx 18.85 \text{ in} \end{aligned}$$

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(3)^2 \\ &= 9\pi \text{ in}^2 \\ &\approx 28.27 \text{ in}^2 \end{aligned}$$

Example

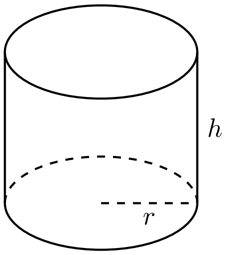
Find the area of a circle whose circumference is 24π cm.

To find the area, we first find the radius.

$$\begin{aligned} C &= 2\pi r \\ 2\pi r &= 24\pi \\ \frac{2\pi r}{2\pi} &= \frac{24\pi}{2\pi} \\ r &= 12 \text{ cm} \end{aligned}$$

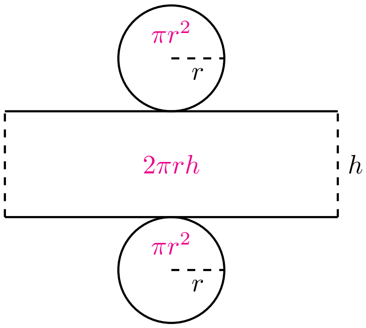
$$\begin{aligned} A &= \pi r^2 \\ &= \pi(12)^2 \\ &= 144\pi \text{ cm}^2 \\ &\approx 452.39 \text{ cm}^2 \end{aligned}$$

8.5 Cylinder Surface Area



A cylinder is a 3D shape similar to a prism[1], with circles for the bases and a single curved rectangle for the lateral surface.

We can still use the surface area formula for prisms, $S = 2B + Ph$. Since the bases are circles with radius r, we have the base area $B = \pi r^2$ and perimeter (circumference) $P = 2\pi r$.

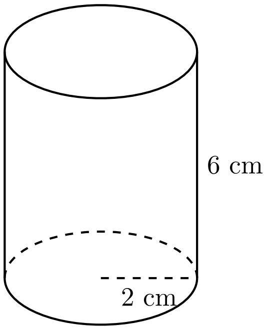


The SURFACE AREA of a CYLINDER with base radius r and height h

$$S = 2\pi r^2 + 2\pi rh$$

Example

Find the surface area of the cylinder shown.



$$r = 2 \qquad h = 6$$

$$\begin{aligned} S &= 2\pi r^2 + 2\pi rh \\ &= 2\pi(2)^2 + 2\pi(2)(6) \\ &= 8\pi + 24\pi \\ &= 32\pi \text{ cm}^2 \\ &\approx 100.53 \text{ cm}^2 \end{aligned}$$

Example

A cylindrical tin cup has a diameter of 7 cm and a height of 10 cm. What is the area of the tin which forms the cup?

A cup only has one base, not two, so we need to adjust the surface area formula.

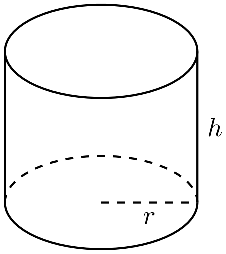
$$r = 3.5$$
$$h = 10$$

$$\begin{aligned} S &= \pi r^2 + 2\pi rh \\ &= \pi(3.5)^2 + 2\pi(3.5)(10) \\ &= 258.40 \text{ cm}^2 \end{aligned}$$

---

[1] Technically, a prism is a type of *polyhedron*, which means all the faces are flat polygons with straight edges. Because a circle is not a polygon, a cylinder is not a polyhedron, which means it can't be a prism.

8.6 Cylinder Volume



Recall that the volume of prism with base area B and height h is $V = Bh$. A cylinder is similar enough to a prism that this rule still holds.

We know that the base of a cylinder is a circle, and if its radius is r its base area is $B = \pi r^2$. By substituting B, we get the formula for the volume of a cylinder.

The VOLUME of a CYLINDER with base radius r and height h

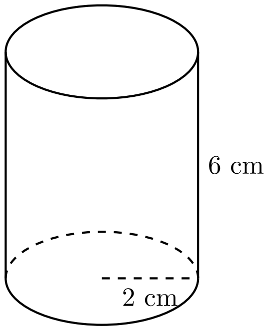
$$V = \pi r^2 h$$

Example

Find the volume of the cylinder shown.

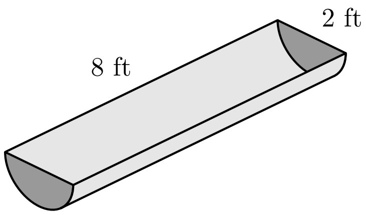
$r = 2$
$h = 6$

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2)^2(6) \\ &= 24\pi \text{ cm}^3 \\ &\approx 75.40 \text{ cm}^3 \end{aligned}$$



Example

Find the capacity of the water trough shown.



The trough is the shape of half of a cylinder.

$r = 1$ ft
$h = 8$ ft

$$\begin{aligned} V &= \tfrac{1}{2} \cdot \pi r^2 h \\ &= \tfrac{1}{2} \cdot \pi(1)^2(8) \\ &= 4\pi \text{ ft}^3 \qquad \approx 12.57 \text{ ft}^3 \end{aligned}$$

9.1 Measures of Central Tendency

A statistic is a single measure which summarizes a characteristic of a collection of data.

A measure of central tendency is a statistic which aims to represent a typical value, or the center, of the data.

MEASURES OF CENTRAL TENDENCY

The mean of a set of data is the sum of the data values divided by the count (the number) of data values.

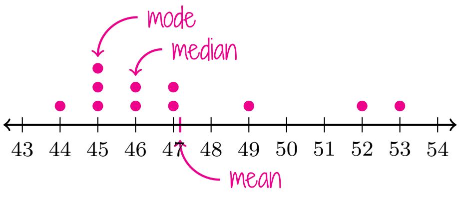
The median of a set of data is the middle value when the data are ordered (for an odd count), or the mean of the two middle values (for an even count).

The mode of a set of data is most frequent value in the data.

Another useful statistic is range, which is a measure of the spread of the data, instead of the center. It is the difference between the largest and smallest values.

Example

Complete the dot plot and calculate the mean, median, mode and range for the data:
46, 44, 47, 53, 45, 52, 45, 47, 49, 46, 45.



The mode is 45 as it appears most often.

44, 45, 45, 45, 46, $\boxed{46,}$ 47, 47, 49, 52, 53.

The median is 46.

$$\frac{46+44+47+53+45+52+45+47+49+46+45}{11} = \frac{519}{11} = 47.18$$

The mean is 47.18

The range is $53 - 44 = 9$

Example

In the first five basketball games of the season, Alex scores 8, 13, 6, 4 and 7 points. What are his mean and median scores?

Mean is $\frac{8+13+6+4+7}{5} = \frac{38}{5} = 7.6$ points.

Median is 7 points: 4, 6, $\boxed{7,}$ 8, 13.

In the sixth game, Alex scores 22 points. How does this change his mean and median scores?

Mean is $\frac{8+13+6+4+7+22}{6} = \frac{60}{6} = 10$ points.

Median is 7.5 points: 4, 6, $\boxed{7,\ 8,}$ 13, 22.

Did including 22 in the data have a bigger effect on the mean or the median? Why?

The mean increased a lot because 22 is quite a bit larger than the mean. The median only increased slightly.

Example

Karissa keeps track of the number of miles she runs each week. Over the last four weeks, she ran 6, 4, 8 and 6 miles respectively.

What is the mean distance Karissa ran each week? What is the median distance?

Mean is $\frac{6+4+8+6}{4} = \frac{24}{4} = 6$ miles.

Ordered data: 4, 6, 6, 8. Median is 6 miles.

Without knowing how many miles Karissa runs in the fifth week, what could possibly happen to the mean and median?

The mean could go up or down, depending on whether her distance in the fifth week is more or less than 6 miles. If that value is close to 6 miles, the change will be small, if it's far from 6 miles, the change will be bigger.

The median will still be 6 miles no matter what happens, as the middle value be 6.

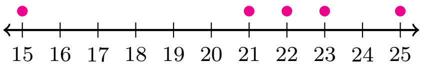
9.2 Outliers

An outlier is a value in a data set whose value is outside the range of values which could be expected from the rest of the data. This typically means outliers are much smaller or larger than the rest of the data.

Outliers need to be carefully investigated, as they are sometimes the result of errors. If an outlier exists, it's a good idea to find a reason its value doesn't fit the rest of the data.

Example

Complete the dot plot, and use it to identify any outliers for the following data:
22, 23, 25, 22, 15, 21



15 is an outlier

Find the mean and median of the data.

Mean: $\frac{22+23+24+22+15+21}{6} = \frac{127}{6} = 21.17$

Median is 22: 15, 21, $\boxed{22, 22}$, 23, 24

Find the mean and median with any outliers removed.

Mean: $\frac{22+23+24+22+21}{6} = \frac{112}{5} = 22.4$

Median is 22: 15, 21, $\boxed{22}$, 23, 24

In general:

- Outliers can have a large effect on the mean.
- Outliers usually have a small effect, or even no effect, on the median.

9.3 Scatterplots and Lines of Best Fit

In statistics, a variable is a characteristic of a person or thing, which can have different values for each person or thing. The data for each person or thing is called an observation. Bivariate data consists of observations of two variables.

A scatterplot is a plot which uses a coordinate plane to represent bivariate data, with a variable on each axis. Each observation is plotted as a point on the plane.

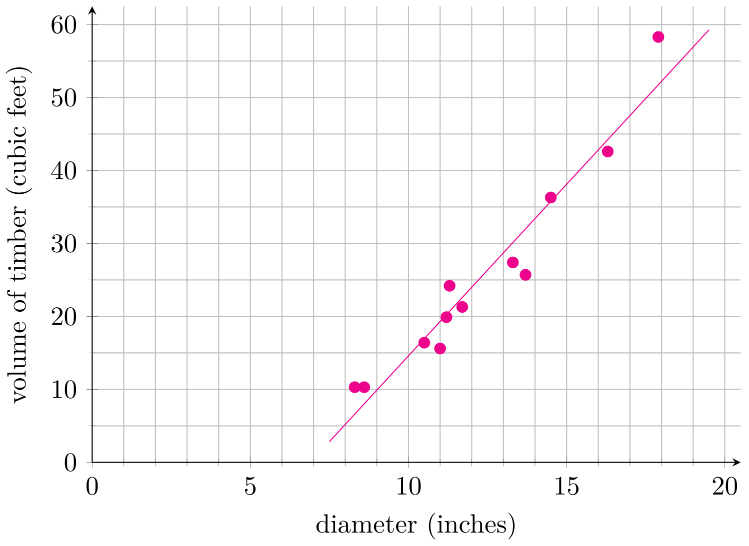
Scatterplots should always include an appropriate title, and labels on each axis with appropriate units.

Example

The table shows the diameter (in inches) and the volume (in cubic feet) of a selection of black cherry trees[1]. Represent the data on the coordinate plane as a scatterplot.

diameter (in)	volume (ft^3)
16.3	42.6
10.5	16.4
11.0	15.6
8.3	10.3
8.6	10.3
14.5	36.3
11.3	24.2
11.7	21.3
13.3	27.4
13.7	25.7
17.9	58.3
11.2	19.9

Diameter and Volume of Black Cherry Trees



A line of best fit is a line we draw on a scatterplot so that it is as close as possible to each of the points on the scatterplot. The line shows the general trend of the data.

In statistics, a model is a function which approximates the relationship between variables. The line of best fit represents a linear model for our two variables.

For now we'll visually estimate the line of best fit. In high school, you'll use software or a calculator to do this more precisely.

Example

1. Draw the line of best fit for the previous scatterplot.
2. Estimate the volume of a black cherry tree with a diameter of 17 inches.

   47.5 ft^3

3. Estimate the rate of change of the volume of a black cherry tree with respect to its diameter.

   The slope of the line of best fit is approximately 4.7. This means a 1 in increase in diameter corresponds to a 4.7 ft^3 increase in volume.

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[1] This is a subset of a dataset available in R, a programming language used by many statisticians. `https://search.r-project.org/R/refmans/datasets/html/trees.html`

10.1 Probabilities and Prediction

An experiment is a random phenomenon whose outcome is unknown until it occurs. The sample space of an experiment is the set of all of its possible outcomes.

Example

State the sample space for each of the following.

1. The side shown on a flipped coin. {heads, tails}
2. The value rolled on a standard 6-sided die. $\{1, 2, 3, 4, 5, 6\}$

An event is a subset of the sample space, or a collection of outcomes.

The probability of an event is a number between 0 and 1 inclusively which indicates how likely an experiment is to produce the event. Probabilities can be written as percentages, fractions, or decimals.

If $P(A) = 0$, then event A is impossible.

If $P(A) = 1$, then event A is certain.

If $P(A) = 0.5$, then event A is equally likely to occur or not occur.

Example

A fair coin is flipped. What is the probability of each of the following events?

- A: The coin lands heads up. $P(A) = 0.5$
- B: The coin lands tails up. $P(B) = 0.5$
- C: The coin lands either heads or tails up. $P(C) = 1$
- D: The coin turns into a pony. $P(D) = 0$

PROBABILITY of event A in sample space S with equally likely outcomes

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{number of outcomes in } A}{\text{number of outcomes in the sample space}}$$

Example

What is the probability that the value rolled on a 10-sided die is a prime number?

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 3, 5, 7\}$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{10} = 0.4$$

Is the number more likely or less likely to be prime than not prime?

Less likely, as the probability that the number is prime is less than 0.5.

Example

Two dice are rolled. Complete the table showing the sums of the possible dice rolls. Find the probabilities of the following events:

A: The sum of the two dice is 4.

$$P(A) = \frac{3}{36} = \frac{1}{12}$$

B: The sum of the two dice is a multiple of 5.

$$P(B) = \frac{7}{36}$$

Which sum is most likely to be rolled? What is its probability?

$$P(7) = \frac{6}{36} = \frac{1}{6}$$

		First Die Roll					
		1	2	3	4	5	6
Second Die Roll	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

10.2 Experimental Probability

Often it isn't possible to calculate the probabilities of events. Instead, we can repeat an experiment many times, and use the outcomes to estimate the probabilities of the events. These estimates are called experimental probabilities.

A trial is an individual performance of an experiment. Increasing the number of trials improves our confidence that the experimental probability is close to the true probability.

The EXPERIMENTAL (ESTIMATED) PROBABILITY of event A

$$P(A) = \frac{\text{number of trials resulting in } A}{\text{total number of trials}}$$

Example

Janey is the goalkeeper for her soccer team. She keeps records for all the penalty kicks she defends. Last season, 21 penalty goals were scored against her, while she saved 9 attempts.

Estimate the percentage probability that Janey will save the next penalty kick against her.

Let event A be that Janey saves the next penalty kick.

$$P(A) = \frac{9}{30} = \frac{3}{10} = 30\%$$

Example

A bag contains an unknown mixture of colored marbles. One at a time, marbles are drawn randomly, then placed back in the bag. There were 7 blue marbles, 3 red marbles and 2 green marbles drawn.

Estimate the probability that the next marble is red.

$P(\text{red}) = \frac{3}{12} = \frac{1}{4}$

Estimate the probability that the next marble is yellow.

$P(\text{yellow}) = \frac{0}{12} = 0$

Are there yellow marbles in the bag?

It seems unlikely, but we don't know. Just because our trials didn't find any doesn't mean for certain that there are no yellow marbles.

There are 48 marbles in the bag. Estimate the number of green marbles.

$P(\text{green}) = \frac{2}{12} = \frac{1}{6}$

$n(\text{green}) = 48 \cdot \frac{1}{6} = 8$

10.3 Independent and Dependent Events

Consider the probabilities of two (or possibly more) events. The events are called independent if the occurrence of one event does not change the probability of the other.

Events that are not independent are called dependent events.

Example

Two dice are rolled. Let A be the event that the first die is even. Let B be the event that the second die is six.

What is $P(B)$? $P(B) = \frac{1}{6}$

Suppose we know that A occurs (the first die is even). What is $P(B)$ now? $P(B) = \frac{1}{6}$

Are A and B independent?

Yes, because the occurence of A did not change the probability of B.

Example

One die is rolled. Let C be the event that the die is odd. Let D be the event that the die is five.

What is $P(D)$? $P(D) = \frac{1}{6}$

Suppose we know that C occurs (the die is odd). What is $P(D)$ now? $P(D) = \frac{1}{3}$

Are C and D independent?

No, because the occurrence of C changed the probability of D.

The PROBABILITY of two INDEPENDENT EVENTS A and B both occuring

is the product **of their individual probabilities**

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example

Consider the two dice from the first example. What is the probability that the first is even at the same time the second is six?

$$\begin{aligned} P(A \text{ and } B) &= P(A) \cdot P(B) \\ &= \frac{1}{2} \cdot \frac{1}{6} \\ &= \frac{1}{12} \end{aligned}$$

10.4 Sampling Techniques

When using data, a population is a collection of all the people or things in which we're interested. In practice, it may be too difficult to collect data from the entire population. Instead, we only collect data from a sample, which is a subset of the population.

Example

Identify the population and sample in each of the following.

1. A frozen foods factory chooses 10 pizzas to heat and test.

   Population: All the frozen pizzas made in the factory.
   Sample: The 10 tested frozen pizzas.

2. A pollster phones 500 voters to ask who they intend to vote for in the next election for Oklahoma governor.

   Population: All voters in Oklahoma.
   Sample: The 500 voters who were phoned.

A good sample should be representative of the population, which means the data produces similar results. This means sample should be as large as is practical. This also means is should be a random sample, meaning the members of the sample are chosen from the population randomly. A sample which is not representative of the population is called a biased sample, or a limited sample.

Example

To determine the fitness of students at a middle school, 20 students are chosen to each run a mile. Are the following samples random or biased?

1. The principal makes an announcement asking for 20 volunteers.

   Biased, students who like running are more likely to volunteer.

2. 20 names are drawn from a hat with the names of every student in the school.

   Random.

3. Each student is assigned a number, and a computer uses a random number generator to choose 20 numbers.

   Random.

4. Mrs. Henley's sixth grade science class, which has 20 students.

   Biased, sixth grade students are not representative of the whole school.